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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. II.

FROM NOVEMBER, 1866, TO NOVEMBER, 1869.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. II.

November 8th, 1866.

ANNUAL GENERAL MEETING, held at Burlington House.

Prof. CAYLEY, Vice-President, in the Chair.

The Reports of the Secretaries and Treasurer were read and adopted.

From the Report of the Secretaries, it appeared that the Society had, since the last General Meeting, held Jan. 15, 1866, increased in number from sixty-nine to ninety-four members.

The communications which had been made to the Society in this interval were as follows:—

"On Polygons of an even number of Sides:" by Mr. Thos. Cotterill.

"On a Proof that every Function has a Root:" by Prof. De Morgan.

"On Various Properties of Confocal Cartesian Ovals:" by Mr. Crofton.

"On the Centres of Algebraical Curves and Surfaces:" by Mr. Roberts.

"On an Addition to Poinsot's Ellipsoidal Mode of representing the Motion of a Rigid Body turning freely round a Fixed Point, whereby the time may be made to register itself mechanically:" by Prof. Sylvester.

"On the Correspondence of Two Points on a Curve:" by Prof. Cayley.

"On the Multiplication of four Theta Functions:" by Prof. H. J. S. Smith.

"On a Problem of Probabilities connected with Parliamentary Elections:" by Mr. Spottiswoode.

"On an Involution System of Circular Cubics, and description of

the curve by points, when the double focus is on the curve:" by Mr. Thos. Cotterill.

Besides these, many lesser communications had been made.

The following Mathematical Journals had been subscribed for:— "Crelle's Journal," the "Nouvelles Annales de Mathématiques," the "Giornale di Matematiche" of Naples, the "Quarterly Journal of Mathematics," the "Oxford, Cambridge, and Dublin Messenger of Mathematics," and the "Mathematical Reprint of the Educational Times." The monthly numbers of the latter journal had been presented to the Society by Mr. Ranyard.

The Proceedings of the Society had been sent to the following Societies and Institutions, viz., the Royal Societies of England and Edinburgh, the Royal Irish Academy, the Académie des Sciences of Paris, the Philosophical Society of Cambridge, the Philosophical Society of Manchester, the Institute of Actuaries, and the British Museum.

In return for these, the Society had received the Proceedings of the Royal Society, and the Journal of the Institute of Actuaries.

By the kindness of the Chemical Society in lending their Rooms, the Society had been enabled to hold their Meetings at Burlington House, where they now met for the first time.

Considerable alterations had been made in the Rules of the Society; among others, a Rule had been introduced creating a class of Foreign Members, to be chosen from among Foreign Mathematicians of eminence residing abroad.

A valuable present of books, consisting of nearly all the Mathematical Library of the late Sir John Lubbock, had been received from his son, the present Sir John Lubbock.

The following Officers and Council were elected to serve during the ensuing year:—President, Prof. Sylvester; Vice-Presidents, Profs. Cayley and De Morgan, and Mr. Spottiswoode; Treasurer, Prof. Hirst; Secretaries, Messrs. G. C. De Morgan and Jenkins; other Members of Council, Messrs. Adler, Thos. Cotterill, Crofton, Ellis, Ingleby, Kisch, Nesbitt, Roberts, and Tucker.

After the conclusion of business, Prof. Sylvester explained a method of remembering Gauss's formulæ for Spherical Triangles. Mr. Thos. Cotterill communicated a simple property of a Cubic Curve and the Polar Conic of a Point upon it.

November 22nd, 1866.

Prof. SYLVESTER, President, in the Chair.

Richard Wormell, Esq., M.A., was elected a Member.

A vote of thanks was passed to Prof. De Morgan for his services as President of the Society during the first two years of its existence; to

the President and Council of the Chemical Society for the use of their rooms at Burlington House; and to Mr. Spottiswoode for a present of books.

The following Paper was read:—

On the General Theory of Anharmonics. By W. K. Clifford, Scholar of Trinity College, Cambridge.

1. The theory of Anharmonics on the straight line may be stated in the following symmetrical form:—

(i.) There is an identical relation connecting the distances of four points, 1, 2, 3, 4, on a right line, viz.,

$$12.34+13.42+14.23 \equiv 0.$$

(ii.) The ratios of the three terms in this identity are not altered by projection or integral linear transformation. There is a corresponding theory of four straight lines meeting in a point.

2. In applying this theory to geometry of two dimensions, we meet with this third proposition:—

(iii.) If a straight line meet four fixed straight lines, so that the distances of the points which they determine on it satisfy a relation of the form

$$\lambda.12.34+\mu.13.42+\nu.14.23 = 0,$$

then the envelope of the line is of the second class, touching the four given lines. There is, of course, a correlative proposition on the locus of a point subtended by four points in a given manner. The propositions (i.), (ii.), (iii.), each including its converse and correlative propositions, may be regarded as constituting the entire theory of anharmonics in geometry of one dimension, and its application to geometry of two dimensions.

3. I proceed to state the theory of anharmonics in geometry of two dimensions.

(i.) There are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123.456+124.563+125.634+126.345 \equiv 0,$$

with five others obtained from this by permutation.

(ii.) The ratios of the terms in these identities are not altered by projection or integral linear transformation. Under each of these propositions are included *three* correlatives, to explain which I must introduce three new definitions:

Def. 1st. The *projector* of a plane triangle is the square of the area divided by the continued product of the sides.

Def. 2nd. In a solid angle considered as determined by three concurrent straight lines, the sine of the angle between the first line and the plane of the other two, multiplied by the sine of the angle between

the other two lines, is a symmetrical quantity in respect of the three lines, and is called the *sine* of the three lines.

Def. 3rd. In a solid angle considered as determined by three planes, the sine of the angle between the first plane and the intersection of the other two, multiplied by the sine of the angle between the other two planes, is a symmetrical quantity in respect of the three planes, and is called the *sine* of the three planes.

5. It is convenient for several purposes to use the word "distance" as including all these notions: thus I shall speak of the *distance* of two lines in a plane, meaning the sine of the angle between them;

the distance of three points is the area of their triangle;

" " of three straight lines in a plane, the projector of their triangle;

" " of three planes, the sine of the planes;

" " of three concurrent lines in space, the sine of the lines.

And I shall have occasion afterwards to define the distance of four points and of four planes in space. By means of these definitions, the propositions (i.), (ii.) may be interpreted in four different ways, corresponding to the four aspects of bi-dimensional extension: the symbol 123 being always understood to mean the *distance* of the three things considered.

6. The propositions (i.), (ii.), including their converse and correlative propositions, constitute the entire theory of anharmonics in geometry of two dimensions. To apply this theory to geometry of three dimensions, I state the following proposition, which has only *one* correlative:

(iii.) If a plane meet six fixed planes so that the distances of the lines they determine upon it satisfy a relation of the form

$$\lambda . 123 . 456 + \mu . 124 . 563 + \nu . 156 . 234 + \ldots\ldots = 0,$$

then the envelope of the plane is of the second class, touching the six given planes.

7. The theory of anharmonics in three dimensions is so entirely analogous to the two former theories, that it wants no further discussion. The distance of four points is the volume of their tetrahedron, and the distance of four planes is the cube of the volume divided by the product of the areas of the faces.

8. Two finite lines $11'$, $22'$, measured on the same straight line, are said to be *harmonically situate* when

$$12 . 1'2' + 12' . 1'2 = 0;$$

and when this is the case, if the two pairs of points be represented by the equations $U = 0$, $V = 0$, there is an invariant relation connecting the quadrics U, V which may be denoted by

$$\square (U, V) = 0;$$

the notation indicating that if we expand the discriminant of $\lambda U + \mu V$, or

$$\square\ (\lambda U + \mu V),$$

then the coefficient of $\lambda\mu$ in the expansion vanishes.

There is a similar relation between two angles having a common vertex; when this relation holds, I say that the two convertical angles are harmonically situate. I also speak of an harmonic pair of angles, or of an harmonic pair of finite lines, or lengths; meaning in this case convertical angles, or collinear lengths.

9. This being so, it is known that two lengths (I use the word *lenyth* to denote a pair of points), anyhow placed in a plane, determine a conic passing through their ends, which conic is the locus of points which the lengths subtend harmonically, that is, in a pair of harmonic angles. I call this the harmonic conic of the two lengths. This (with the correlative propositions) completes the theory of harmonics in one dimension, and its application to two dimensions. I now come to consider harmonics in two dimensions.

10. If the harmonic conic of two lengths $11'$, $22'$ divides harmonically a third length $33'$, then the relation between the three lengths is symmetrical, and I say that the three lengths are *harmonically situate* in the plane. The following relation subsists among their distances, viz., $123 \cdot 1'2'3' + 1'23 \cdot 12'3' + 12'3 \cdot 1'23' + 123' \cdot 1'2'3 = 0$.

And if each length, or point-pair, be considered as a degenerate conic, so that the equations to the three point-pairs are $U, V, W = 0$, then, when the three lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, which may be denoted by

$$\square\ (U, V, W) = 0;$$

$\square\ (U, V, W)$ denoting the coefficient of $\lambda\mu\nu$ in the expansion of

$$\text{Discriminant of } (\lambda U + \mu V + \nu W).$$

It is obvious that a similar relation may subsist among three angles in a plane, three pairs of lines through a point in space, or three pairs of planes through a point in space.

11. Three lengths anyhow placed in space determine a quadric surface passing through their ends, which surface is the locus of points which the lengths subtend harmonically, that is, in a triad of harmonic angles. I call this the harmonic quadric of the three lengths.

12. If the harmonic quadric of three lengths $11'$, $22'$, $33'$ divides harmonically a fourth length $44'$, then the relation between the four lengths is symmetrical, and I say that the four lengths are *harmonically situate* in space. The following relation subsists among their distances, viz.,

$$\Sigma \cdot 1234 \cdot 1'2'3'4' = 0 \text{ (eight terms)}.$$

And if each length, or point-pair, be considered as a degenerate quadric, so that the equations to the four point-pairs are $U=0$, $V=0$, $W=0$, $T=0$, then, when the four lengths are harmonically situate,

there is an invariant relation connecting the quadrics U, V, W, T, which may be denoted by

$$\square\,(U, V, W, T) = 0;$$

$\square\,(U, V, W, T)$ denoting the coefficient of $\lambda\mu\nu\rho$ in the expansion of

Discriminant of $(\lambda U + \mu V + \nu W + \rho T)$.

It is obvious that a similar relation may subsist among four pairs of planes.

13. It only remains to explain the meaning of the conditions

$$\square\,(U, V, W) = 0,$$
$$\square\,(U, V, W, T) = 0,$$

when the quadrics do *not* break up into factors. Two conics U, V determine a harmonic conic F, locus of points which they subtend in a pair of harmonic angles. If a triangle self-conjugate of F can be inscribed in W, then the relation between UVW is symmetrical, and $\square\,(U, V, W) = 0$. The conics may then be spoken of as three mutually harmonic conics; a similar relation may hold between three convertical cones. Thus, in fact, three quadric *surfaces* U, V, W determine a harmonic quadric F, locus of points which they subtend in three harmonic cones. If a tetrahedron self-conjugate of F can be inscribed in T, then the relation between U, V, W, T is symmetrical, and $\square\,(U, V, W, T) = 0$.

Prof. Sylvester communicated a development for the n^{th} power of a logarithm, with some theorems depending upon it.

A discussion took place with regard to the reflection of a rainbow in still water.

Mr. Tucker communicated a solution of the following problem:—"Given a pair of conjugate diameters of an ellipse, to find any number of points on the curve."

December 13th, 1866.

Prof. SYLVESTER, President, in the Chair.

Rev. D. Thomas, M.A., and Messrs. W. H. Besant, M.A., W. H. Corfield, B.A., J. D. Davenport, M.A., and H. Macneile, B.A., were elected Members.

Prof. Cayley exhibited and explained some geometrical drawings. Thinking that the information might be convenient for persons wishing to make similar drawings, he noticed that the paper used was a tinted drawing paper, made in continuous lengths up to 24 yards, and of the breadth of about 56 inches;* the half-breadth being therefore sufficient

* Sold at Messrs. Lechertier-Barbe's, Regent Street, at 6*d*. per yard, or 9*s*. the piece.

for ordinary figures, and the paper being of a good quality and taking colour very readily. Among the drawings was one of the conics through four points forming a convex quadrangle. The plane is here divided into regions by the lines joining each of the six pairs of points, and by the two parabolas through the four points; and the regions being distinguished by different colours, the general form of the conics of the system is very clearly seen. (Prof. Cayley remarked that it would be interesting to make the figures of other systems of conics satisfying four conditions; and in particular for the remaining elementary systems of conics, where the conics pass through a number 3, 2, 1 or 0 of points and touch a number 1, 2, 3 or 4 of lines: the construction of some of these figures is, however, practically a great deal more difficult.) Other figures related to Cartesians and Bicircular Quartics. One of these was a figure of a system of triconfocal Cartesians; and derived from this by inversion in regard to a circle, there was a figure of a system of quadriconfocal bicircular quartics: in the assumed position of the inverting circle, each quartic consists (like the Cartesian which gives rise to it) of an exterior and an interior continuous curve, and the general aspect of the figure is that of a distortion of the original figure of the Cartesians. Another figure was that of the bicircular quartic, for which the algebraical sum of the distances of a point thereof from three given foci is $= 0$ (this was selected for facility of construction, by the intersections of circles and confocal conics). The quartic consists of two equal and symmetrically situated pear-shaped curves, exterior to each other, and including the one of them two of the three given foci, the other of them the third given focus, and a fourth focus lying in a circle with the given foci: by inversion in regard to a circle having its centre at a focus, the two pear-shaped curves became respectively the exterior and the interior ovals of a Cartesian. There was also a figure of the two circular cubics, having for foci four given points on a circle: and a figure (coloured in regions) in preparation for the construction of the analogous sextic curve derived from four given points not in a circle.

The following communication was read: "On the Development of a certain class of Functions:" by Mr. G. C. De Morgan.

Prof. Hirst presented to the Society some Models of Surfaces of the fourth order and class, illustrative of Prof. Plücker's new Theory of Complexes.

The thanks of the Meeting were returned to Prof. Hirst for this present.

January 24th, 1867.

WM. SPOTTISWOODE, Esq., Vice-President, in the Chair.

Messrs. J. O. Corrie, B.A., J. J. Downes, and R. Moon, M.A., were elected Members.

A vote of thanks was passed to Dr. Sharpey for the use of his shelves at University College for the books belonging to the Society.

The following communications were made: "On some Cases of Failure of Maclaurin's Development,"* by Prof. De Morgan, and

On the Order of certain Systems of Algebraical Equations.

By Samuel Roberts, M.A.

1. A simultaneous system of $k-s$ equations in k independent variables, combined with s linear equations in the same variables, determines a certain number of simultaneous solutions. This number is called the Order of the system of $k-s$ equations, and is in fact the product of their degrees. But the system of $k-s$ equations may be only implied as the result of eliminating another set of variables with respect to which the given system is explicit. Thus we may have a system of $l+t$ equations in l independent variables, the coefficients being functions of k independent variables. We may consider such a system as representing a system of t equations in k variables. If we could always and readily obtain this secondary system explicitly, there would be no difficulty in obtaining the order. But this is not the case. The usual processes of elimination introduce extraneous solutions, which have to be allowed for; and moreover, it may be that no explicit system with integer indices will represent exclusively the implied system. For example, a prime number of points in a plane cannot be represented exclusively by the intersections of two curves.

One purpose of the present paper is to determine the net orders of such implied systems as I have described, or simply the orders of the systems. I confine myself, however, to binary systems in regard to the variables to be eliminated.

Again, with respect to the variables to be eliminated, a given system may be doubly, trebly, &c., coexistent; in other words, may be simultaneously solved by more than one set of values. In this case also extraneous solutions present themselves, and the implied system cannot be usually given in an explicit form. I purpose to treat some systems conditioned in this way. The necessity of brevity will allow me to give only so much detail as may be required to make intelligible the method employed. Fortunately, I am able to refer the reader to Dr.

* The principal part of this communication is given as a postscript to a paper in the Transactions of the Camb. Phil. Society, Vol. XI., Part II., "On the Root of any Function; and on Neutral Series."

Salmon's "Lessons introductory to the Modern Higher Algebra," 2nd edition, Lesson xviii., in which the present subject is fully explained.

2. For our purpose, it is permissible to suppose that the functions which form the coefficients of a given system are conditioned in any manner we please, which does not affect their order. The order of the system depends on the degrees in which the variables not eliminated enter the coefficients, and not at all on their special forms. If, therefore, we have a quantic in x and y of the degree n, the coefficients of x^n, $x^{n-1}y$, $x^{n-2}y^2$, &c., being of the orders ν, $\nu+\alpha$, $\nu+2\alpha$, &c., in a second set of variables, it may be adequately represented by the n factors

$$(\mathrm{A}x+\mathrm{B}y)\,(\mathrm{A}'x+\mathrm{B}'y)\,(\mathrm{A}''x+\mathrm{B}''y)\ \&c.,$$

where A, A′, A″, &c., B, B′, B″, &c., are of the orders α, α', α'', &c., $\alpha+\alpha$, $\alpha'+\alpha$, $\alpha''+\alpha$, &c., and $\Sigma\alpha=\nu$. We may distribute the total order ν into what parts we please; it is usually convenient to put $\alpha=\alpha'=\&c.=\frac{\nu}{n}$. When this quantity is fractional, I shall still use the term "order" for results to which it would be strictly applicable if $\frac{\nu}{n}$ were integer. It is not, however, necessary to embarrass our processes by discussing the meaning of a fractional order. We know that the result which we seek is, with reference to the above quantic, a function of ν or $\Sigma\alpha$, and the substitution of $\frac{\nu}{n}$ for α, α', α'', &c. may be made before or after our operations, without affecting the result.

3. In the case of linear systems, the notation of determinants is commonly employed. Thus

$$\begin{vmatrix} a & b \\ a' & b' \\ a'' & b'' \end{vmatrix} = 0$$

represents two conditions, involving the functions indicated by the letters.

Suppose we take two systems, formed of the 1st and 2nd and of the 1st and 3rd horizontal lines. Then, if a, a', a'' are of the orders λ, μ, ν, and b, b', b'' are of the orders $\lambda+\alpha$, $\mu+\alpha$, $\nu+\alpha$, the resulting conditions are of the order $(\lambda+\mu+\alpha)\,(\lambda+\nu+\alpha)$.
But they are satisfied by the irrelevant system $(ab)=0$, which does not satisfy the given system, and we must deduct the order $\lambda\,(\lambda+\alpha)$. The net result is therefore

$$\Sigma\lambda\mu+\alpha\Sigma\lambda+\alpha^2.$$

Again, if a''', b''' are of the orders π, $\pi+\alpha$, and the given system, representing three conditions, is

$$\begin{vmatrix} a & b \\ a' & b' \\ a'' & b'' \\ a''' & b''' \end{vmatrix} = 0,$$

we take the two systems made up of the 1st, 2nd, and 3rd, and of the 1st and 4th horizontal lines. The order of the two systems is

$$(\lambda+\pi+a)\,(\Sigma\lambda\mu+a\Sigma\lambda+a^2),$$

π not appearing in the second factor. But we have included the irrelevant system

$$\begin{vmatrix} a & \\ & b \\ a' & b' \\ a'' & b'' \end{vmatrix} = 0,$$

which satisfies the two systems selected, but does not satisfy the given system; that is to say, we must deduct $\lambda\,(\lambda+a)\,(\mu+\nu+a)$, and our net result is

$$\Sigma\lambda\mu\nu+a\Sigma\lambda\mu+a^2\Sigma\lambda+a^3.$$

Proceeding in this way, we are led to the conclusion that, for a system of $k+1$ simple equations, having coefficients whose orders follow the same law, the order is

$$a^k+a^{k-1}.\,\mathrm{P}+a^{k-2}.\,\mathrm{Q}+\&\mathrm{c.}\ \ldots\ldots\ldots\ldots\ldots\ (\mathrm{A}),$$

where P is the sum of λ, μ, ν, &c., Q is the sum of their products, two and two together; and so on, to $k+1$ terms.

4. Assume, then, that the formula (A) holds for any such system of less than $k+2$ equations; and let another similar equation be added, whose coefficients are of the orders σ, $\sigma+a$. Then, taking the original system and the system formed of the 1st and the last added equations, we get an order

$$(\lambda+\sigma+a)\,(a^k+a^{k-1}.\,\mathrm{P}+a^{k-2}.\,\mathrm{Q}+\&\mathrm{c.}).$$

But this includes the order of the irrelevant system

$$\begin{vmatrix} a & \\ & b \\ a' & b' \\ a'' & b'' \\ & \&\mathrm{c.} \\ a^{(k)'} & b^{(k)'} \end{vmatrix} = 0,$$

which satisfies the selected systems, but does not satisfy the given system; and we must deduct the order

$$\lambda\,(\lambda+a)\,(a^{k-1}+a^{k-2}.\,\mathrm{P}'+a^{k-s}.\,\mathrm{Q}'+\&\mathrm{c.}),$$

where P′ is the sum of μ, ν, &c., Q′ is the sum of their products in pairs, &c., λ and σ being omitted in each case. Evidently the coefficient of a^{k+1} in the result is unity, that of a^k is $\Sigma\lambda$, and that of a^{k-1} is $(\lambda+\sigma)\,\mathrm{P}+\mathrm{Q}-\lambda^2-\lambda\mathrm{P}'$, or $\Sigma\lambda\mu$, &c.; and, generally, the coefficient of a^{k-s+1} is the sum of the products of λ, μ, ν, taken s together. Hence the formula (A), which holds for $k=2$, $k=3$, holds good generally.

5. Before proceeding further, the following observation is necessary. If the sets of algebraical conditions

$$\begin{vmatrix} \phi_1 \\ \phi_2 \\ \&c. \\ \phi_k \end{vmatrix} = 0, \quad \begin{vmatrix} \psi_1 \\ \psi_2 \\ \&c. \\ \psi_k \end{vmatrix} = 0, \quad \begin{vmatrix} \chi_1 \\ \chi_2 \\ \&c. \\ \chi_k \end{vmatrix} = 0, \quad \&c.,$$

are such that, while each set refers to and comprises a definite class of the cases of a problem, they together refer to and comprise all its cases, we may take for a system of conditions comprising all the cases

$$\begin{aligned} \phi_1 \cdot \psi_1 \cdot \chi_1 \ldots &= 0, \\ \phi_2 \cdot \psi_2 \cdot \chi_2 \ldots &= 0, \\ \&c. \qquad\quad &\&c. \\ \phi_k \cdot \psi_k \cdot \chi_k \ldots &= 0. \end{aligned}$$

But this system includes a number of systems not agreeing with the original sets (*e.g.*, $\phi_1 = \psi_2 = \ldots = \chi_k = 0$). The orders of all such have to be deducted, and the order of the general system after the deduction becomes simply the sum of the orders of the original sets. In calculating the result, we must take the net order of those sets after the rejection of extraneous solutions, if such there be.

6. The foregoing conclusion may be applied to the case of a system of binary equations (in x, y) of any degree. Let there be $k+1$ such equations of the degrees m_1, m_2, m_3, &c., m_{k+1}, and let the coefficients of x^{m_1}, x^{m_2}, &c., $x^{m_{k+1}}$ be of the orders μ_1, μ_2, &c., μ_{k+1}, while the increment of these orders, as we pass from any term to the next higher in y, is α. We substitute quantics separable into factors of appropriate form. The coefficients of the factors of the 1st quantic are taken to be of the orders $\frac{\mu_1}{m_1}$, $\frac{\mu_1}{m_1} + \alpha$, those of the factors of the 2nd quantic are $\frac{\mu_2}{m_2}$, $\frac{\mu_2}{m_2} + \alpha$, &c. Then, taking a factor from each set, we get the order of this elementary system by the formula (A). The number of ways in which such a system can be formed is $m_1, m_2, \ldots m_{k+1}$; and the order of each system is the same. The order, therefore, of the general system, being the sum of the orders of the elementary systems, is

$$m_1 m_2 \ldots m_{k+1} (\alpha^k + \alpha^{k-1} \mathrm{P} + \alpha^{k-2} \mathrm{Q} + \&c.),$$

where P is the sum of $\frac{\mu_1}{m_1}$, $\frac{\mu_2}{m_2}$, &c., Q is the sum of their products in pairs, &c. This result may be usefully written in the forms

$$\alpha^k m_1 m_2 \ldots m_{k+1} + \alpha^{k-1} \Sigma \mu_1 m_2 \ldots m_{k+1} + \alpha^{k-2} \Sigma \mu_1 \mu_2 m_3 \ldots m_{k+1} + \&c.,$$

and

$$\frac{(\alpha m_1 + \mu_1)(\alpha m_2 + \mu_2) \ldots (\alpha m_{k+1} + \mu_{k+1}) - \mu_1 \mu_2 \ldots \mu_{k+1}}{\alpha} \ldots\ldots (\mathrm{A}').$$

7. I pass now to systems which are supposed to have more than one common root. To economize space, we will first take a system of three equations having two common solutions. A system of two equations can be similarly treated.

Let the equations be of the degrees l, m, n; and let the orders of the coefficients of $x^l, x^{l-1}y$, &c., $x^m, x^{m-1}y$, &c., $x^n, x^{n-1}y$, &c., in the several equations, be $\lambda, \lambda+\alpha$, &c., $\mu, \mu+\alpha$, &c., $\nu, \nu+\alpha$, &c. We separate the quantics into factors. The orders of the coefficients of any factor of the first quantic are taken to be $\frac{\lambda}{l}, \frac{\lambda}{l}+\alpha$, &c., and so with the rest of the quantics.

Now the cases of the problem are of three distinct kinds, according to the number of factors of the same equation involved in the elementary system which we form. Indicating these numbers by figures, the elementary systems are denoted by the columns

$$\begin{matrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{matrix}$$

It is evident that one of the elementary equations must contain two factors.

(1.) We take the equations formed of factors of the given system

$$\begin{aligned} (ax+by)\,(a'x+b'y) &= 0 \\ (cx+dy)\,(c'x+d'y) &= 0 \\ (ex+fy)\,(e'x+f'y) &= 0. \end{aligned}$$

Representing these by

$$\begin{aligned} \mathrm{A}\,.\,\mathrm{B} &= 0 \\ \mathrm{C}\,.\,\mathrm{D} &= 0 \\ \mathrm{E}\,.\,\mathrm{F} &= 0, \end{aligned}$$

we see that there will be two common solutions involving the coefficients of all the factors in the system if

$$\left\{ \begin{matrix} \mathrm{A} \\ \mathrm{C} \\ \mathrm{E} \end{matrix} \right| = 0 \quad \left| \begin{matrix} \mathrm{B} \\ \mathrm{D} \\ \mathrm{F} \end{matrix} \right| = 0 \Bigg\}, \quad \text{or} \quad \left\{ \begin{matrix} \mathrm{A} \\ \mathrm{D} \\ \mathrm{F} \end{matrix} \right| = 0 \quad \left| \begin{matrix} \mathrm{B} \\ \mathrm{C} \\ \mathrm{E} \end{matrix} \right| = 0 \Bigg\},$$

$$\text{or} \quad \left\{ \begin{matrix} \mathrm{A} \\ \mathrm{C} \\ \mathrm{F} \end{matrix} \right| = 0 \quad \left| \begin{matrix} \mathrm{B} \\ \mathrm{D} \\ \mathrm{E} \end{matrix} \right| = 0 \Bigg\}, \quad \text{or} \quad \left\{ \begin{matrix} \mathrm{A} \\ \mathrm{D} \\ \mathrm{E} \end{matrix} \right| = 0 \quad \left| \begin{matrix} \mathrm{B} \\ \mathrm{C} \\ \mathrm{F} \end{matrix} \right| = 0 \Bigg\}.$$

And the order of each of these sub-systems is $\left[\Sigma\frac{\lambda\mu}{lm}+\alpha\Sigma\frac{\lambda}{l}+\alpha^2\right]^2$, so that, taking the sum of the orders, we have, for the order of the elementary system which we are considering,

$$4\left[\Sigma\frac{\lambda\mu}{lm}+\alpha\Sigma\frac{\lambda}{m}+\alpha^2\right]^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

It may be right to remark here, that we have not been determining the order of the conditions that the elementary system may have two common solutions generally, but have assumed that they are due to distinct factors in each equation.

(2.) Again, the system

$$\begin{aligned}(ax+by)(a'x+b'y) &= 0\\ (cx+dy)(c'x+d'y) &= 0\\ ex+fy &= 0,\end{aligned}$$

$\left.\begin{matrix}A.B=0\\ \text{or } C.D=0\\ E=0\end{matrix}\right\}$ will have two common solutions involving the coefficients of all the factors, if

$$\left\{\begin{vmatrix}A & B\\ C & D\\ E & E\end{vmatrix}=0\right\},\quad \text{or}\quad \left\{\begin{vmatrix}A & B\\ D & C\\ E & E\end{vmatrix}=0\right\},$$

provided we exclude in each case the supposition that

$$\begin{vmatrix}A\\ B\\ C\\ D\\ E\end{vmatrix}=0,$$

which is compatible with only one common solution. We can obtain the order of this system by (A′), and find

$$\frac{\lambda^2\mu^2}{l^2m^2}+2\frac{\lambda^2\mu\nu}{l^2mn}+2\frac{\mu^2\lambda\nu}{m^2ln}+\alpha\left[\frac{\lambda^2\nu}{l^2n}+\frac{\mu^2\nu}{m^2n}+2\frac{\lambda^2\mu}{l^2m}+2\frac{\mu^2\lambda}{m^2l}+4\frac{\lambda\mu\nu}{lmn}\right]$$
$$+\alpha^2\left[\frac{\lambda^2}{l^2}+\frac{\mu^2}{m^2}+2\frac{\lambda\nu}{ln}+2\frac{\mu\nu}{mn}+4\frac{\lambda\mu}{lm}\right]+\alpha^3\left[2\frac{\lambda}{l}+2\frac{\mu}{m}+\frac{\nu}{n}\right]+\alpha^4.$$

Subtracting, then, this expression from $\left[\Sigma\frac{\lambda\mu}{lm}+\alpha\Sigma\frac{\lambda}{l}+\alpha^2\right]^2$, and multiplying by 2, we get

$$2\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\lambda}{l}+\frac{\mu}{m}+\alpha\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

In this case, we might have inferred the form directly, since we must evidently have $E=0$ identically, and

$$\left\{\begin{vmatrix}A & B\\ C & D\end{vmatrix}=0\right\},\quad \text{or}\quad \left\{\begin{vmatrix}A & B\\ D & C\end{vmatrix}=0\right\}.$$

(3.) Lastly, the system

$$\begin{aligned}(ax+by)(a'x+b'y) &= 0\\ cx+dy &= 0\\ ex+fy &= 0\end{aligned}$$

gives the order

$$\frac{\mu}{m}\frac{\nu}{n}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

since we must have $c=d=e=f=0$. The numbers by which we must multiply (a), (b), (c), in order to obtain the corresponding terms in the required result, are easily found by the theory of combinations to be

$$\frac{lmn(l-1)(m-1)(n-1)}{(1.2)^3}\quad \frac{lmn(l-1)(m-1)}{(1.2)^2}\quad \frac{lmn(l-1)}{1.2}.$$

Taking into account, then, the terms which may be derived from these by symmetrical interchange of letters, we get, for the required order,

$$\frac{lmn}{2}\left\{\begin{array}{l}(l-1)(m-1)(n-1)\left(\Sigma\frac{\lambda\mu}{lm}+a\Sigma\frac{\lambda}{l}+a^2\right)^2\\ +\Sigma(l-1)(m-1)\frac{\nu}{n}\left(\frac{\nu}{n}+a\right)\left(\frac{\lambda}{l}+\frac{\mu}{m}+a\right)^2\\ +\Sigma(l-1)\left(\frac{\mu\nu}{mn}\right)\left(\frac{\mu}{m}+a\frac{\nu}{n}+a\right)\end{array}\right\}.$$

The same reasoning applies to a system of $k+1$ binary equations, and the result maintains a corresponding form. Thus, if $k=3$, we have, continuing the notation,

$$\frac{lmnp}{2}\left\{\begin{array}{l}(l-1)(m-1)(n-1)(p-1)\left[\Sigma\frac{\lambda\mu\nu}{lmn}+a\Sigma\frac{\lambda\mu}{lm}+a^2\Sigma\frac{\lambda}{l}+a^3\right]^2\\ +\Sigma(l-1)(m-1)(n-1)\frac{\pi}{p}\left(\frac{\pi}{p}+a\right)\left[\frac{\lambda\mu}{m}+\frac{\mu\nu}{mn}+\frac{\nu\lambda}{nl}\right.\\ \qquad\left.+a\left(\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}\right)+a^2\right]^2\\ +\Sigma(l-1)(m-1)\frac{\pi}{p}\left(\frac{\pi}{p}+a\right)\left(\frac{\nu}{n}\right)\left(\frac{\nu}{n}+a\right)\left[\frac{\lambda}{l}+\frac{\mu}{m}+a\right]^2\\ +\Sigma(l-1)\frac{\mu\nu\pi}{mnp}\left(\frac{\mu}{m}+a\right)\left(\frac{\nu}{n}+a\right)\left(\frac{\pi}{p}+a\right)\end{array}\right\}$$

the law of the form being evident and including the case of $k=1$.

8. Let the capital letters A, B, C, &c. still denote simple factors; then I use $\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\\ \&c.\end{matrix}\right|$ to denote the order of conditions that the system

$\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\\ \&c.\end{matrix}\right| = 0$ may have a common solution, and $\left|\begin{matrix}\mathrm{A} & \mathrm{G}\\ \mathrm{B} & \mathrm{H}\\ \mathrm{C} & \mathrm{K}\end{matrix}\right|$ will denote the order of the conditions that $\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\end{matrix}\right| = 0$, $\left|\begin{matrix}\mathrm{G}\\ \mathrm{H}\\ \mathrm{K}\end{matrix}\right| = 0$ may each have a common solution. But if the systems contain a letter in common, I use an asterisk to denote that the common solutions are distinct. For example, the last elementary system of Article 7 is represented by

$$\begin{array}{l}\mathrm{A}.\mathrm{B}=0\\ \mathrm{C}\ \ =0\\ \mathrm{D}\ \ =0,\end{array}$$

and we have

$$\begin{matrix}A\\C\\D\end{matrix}\Big|\begin{matrix}B\\C\\D\end{matrix}\Big|^{*} = \begin{matrix}A\\C\\D\end{matrix}\Big|\begin{matrix}B\\C\\D\end{matrix}\Big| - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| = \frac{\mu}{m}\frac{\nu}{n}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right).$$

9. It is required to find the order of the conditions that two binary equations may have three solutions in common. Let their degrees be m, n; and let the coefficients of x^m, $x^{m-1}y$, &c., in the first equation, be of the orders μ, $\mu+\alpha$, &c., while the corresponding orders of the coefficients of the second equation are ν, $\nu+\alpha$, &c. We separate the quantics into factors, as in previous cases.

(1.) The system $\begin{matrix}A.B.C=0\\D.E.F=0\end{matrix}$ gives us 2.3 systems of the form

$$\begin{matrix}A\\D\end{matrix}\Big| = \begin{matrix}B\\E\end{matrix}\Big| = \begin{matrix}C\\F\end{matrix}\Big| = 0,$$

and the order is

$$2.3\begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\E\end{matrix}\Big|\begin{matrix}C\\F\end{matrix}\Big|, \quad \text{or} \quad 2.3\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)^3.$$

(2.) The system $\begin{matrix}A.B.C=0\\D.E\ \ =0\end{matrix}$ gives us 2.3 sets of the form

$$\begin{matrix}A\\D\end{matrix}\Big| = \begin{matrix}B\\D\end{matrix}\Big| = \begin{matrix}C\\E\end{matrix}\Big| = 0.$$

The order which we require is

$$2.3\left\{\begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\E\end{matrix}\Big|^{*}\right\} = 2.3\left\{\begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\E\end{matrix}\Big| - \begin{matrix}A\\B\\D\end{matrix}\Big|\begin{matrix}C\\E\end{matrix}\Big|\right\}$$

$$= 2.3\left\{\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right).\right.$$

(3.) We have the system $\begin{matrix}A.B.C=0\\D\ \ \ =0\end{matrix}$ and the order is

$$\begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big|^{*} = \begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| - \begin{matrix}A\\B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big|^{*} - \begin{matrix}A\\C\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|^{*} - \begin{matrix}B\\C\\D\end{matrix}\Big|\begin{matrix}A\\D\end{matrix}\Big|^{*} - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|$$

$$= \begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| - \left\{\begin{matrix}A\\B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|\right\} - \left\{\begin{matrix}A\\C\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big| - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|\right\}$$

$$- \left\{\begin{matrix}B\\C\\D\end{matrix}\Big|\begin{matrix}A\\D\end{matrix}\Big| - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|\right\} - \begin{matrix}A\\B\\C\\D\end{matrix}\Big|$$

$$= \begin{matrix}A\\D\end{matrix}\Big|\begin{matrix}B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| - 3\begin{matrix}A\\B\\D\end{matrix}\Big|\begin{matrix}C\\D\end{matrix}\Big| + 2\begin{matrix}A\\B\\C\\D\end{matrix}\Big| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

And, by the rule for simple factors, the order is found to be

$$\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\nu}{n}-\frac{3\mu}{m}-\alpha\right).$$

It is easy to determine, by the theory of combinations, the numerical multipliers of the previous results in the final formula; and taking into account the terms symmetrically formed from these, we get for the required order

$$\frac{mn}{1.2.3}\left\{\begin{array}{l}(m-1)(m-2)(n-1)(n-2)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)^3\\ +3(m-1)(n-1)(n-2)\frac{\mu}{m}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)\\ +3(n-1)(m-1)(m-2)\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)\\ +(m-1)(m-2)\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\nu}{n}-\frac{3\lambda}{n}-\alpha\right)\\ +(n-1)(n-2)\frac{\mu}{m}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\mu}{m}-\frac{3\nu}{n}-\alpha\right)\end{array}\right\},$$

which will be found to agree with Dr. Salmon's result, obtained in a different manner and a different form. (*Geometry of Three Dimensions*, Appendix IV.)

It will be observed that either one or both of the arguments $\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\nu}{n}-\frac{3\nu}{m}-\alpha,\right)\frac{\mu}{m}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\mu}{m}-\frac{3\nu}{n}-\alpha\right)$ must be negative. A formula which expresses an order may of course become negative for particular values of the quantities involved, and these values may be integers. Thus the order of the conditions that a binary equation may have three equal roots is

$$3(l-2)\lambda(\lambda+l\alpha)+l(l-1)(l-2)\alpha^2,$$

which becomes negative for $l=1$. In the present case, we have to find what the formula (d) becomes on the supposition that the factors are of the first degree. We might work out the formula in the general case, when A, B, C, D are of the degrees s, t, u, v, and afterwards make $s=t=u=v=1$, with the same result. A negative order only means that the general form becomes negative for particular values of the quantities involved. This must be borne in mind with reference to the processes of the next example, and indeed generally in the application of the method.

10. It is required to find the order of the conditions that two binary equations may have four common solutions. Let their degrees and the orders of the coefficients be the same as in the last case. We divide into factors as before.

(1.) The system $\begin{matrix} A \,.\, B \,.\, C \,.\, D = 0 \\ E \,.\, F \,.\, G \,.\, H = 0 \end{matrix}$

gives us $2 \,.\, 3 \,.\, 4$ sets

$$\begin{vmatrix} A \\ E \end{vmatrix} = \begin{vmatrix} B \\ F \end{vmatrix} = \begin{vmatrix} C \\ G \end{vmatrix} = \begin{vmatrix} D \\ H \end{vmatrix} = 0;$$

and the order is

$$2 \,.\, 3 \,.\, 4 \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ G \end{vmatrix} \begin{vmatrix} D \\ H \end{vmatrix} = 2 \,.\, 3 \,.\, 4 \left(\frac{\mu}{m} + \frac{\nu}{n} + \alpha \right)^4.$$

(2.) The system $\begin{matrix} A \,.\, B \,.\, C \,.\, D = 0 \\ E \,.\, F \,.\, G \quad\quad = 0 \end{matrix}$

gives us $(2 \,.\, 3)^2$ sets of the form

$$\left[\begin{vmatrix} A \\ E \end{vmatrix} = \begin{vmatrix} B \\ F \end{vmatrix} = \begin{vmatrix} C \\ G \end{vmatrix} = \begin{vmatrix} D \\ G \end{vmatrix} = 0 \right];$$

and the order of each is

$$\begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ G \end{vmatrix} \begin{vmatrix} D \\ G \end{vmatrix}^* = \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ G \end{vmatrix} \begin{vmatrix} D \\ G \end{vmatrix} - \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ D \\ G \end{vmatrix}$$

$$= \frac{\nu}{n} \left(\frac{\nu}{n} + \alpha \right) \left(\frac{\mu}{m} + \frac{\nu}{n} + \alpha \right)^2.$$

(3.) The system $\begin{matrix} A \,.\, B \,.\, C \,.\, D = 0 \\ E \,.\, F \quad\quad = 0 \end{matrix}$

gives us $4 \,.\, 2$ systems of the kind

$$\left[\begin{vmatrix} A \\ E \end{vmatrix} = \begin{vmatrix} B \\ F \end{vmatrix} = \begin{vmatrix} C \\ F \end{vmatrix} = \begin{vmatrix} D \\ F \end{vmatrix} = 0 \right];$$

and the order of each is

$$\begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix}^* = \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ F \end{vmatrix} \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix} - 3 \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix} + 2 \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ C \\ D \\ F \end{vmatrix}$$

$$= \frac{\nu}{n} \left(\frac{\nu}{n} + \alpha \right) \left(\frac{\nu}{n} - \frac{3\mu}{m} - \alpha \right) \left(\frac{\mu}{m} + \frac{\nu}{n} + \alpha \right).$$

But we have also $3 \,.\, 2$ sets of the kind

$$\begin{vmatrix} A \\ E \end{vmatrix} = \begin{vmatrix} B \\ E \end{vmatrix} = \begin{vmatrix} C \\ F \end{vmatrix} = \begin{vmatrix} D \\ F \end{vmatrix} = 0;$$

and the order of each is

$$\begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ E \end{vmatrix} \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix}^* = \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ E \end{vmatrix} \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix} - \left\{ \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ E \end{vmatrix} \begin{vmatrix} C \\ D \\ F \end{vmatrix} - \begin{vmatrix} A \\ B \\ E \end{vmatrix} \begin{vmatrix} C \\ D \\ F \end{vmatrix} \right\}$$

$$- \left\{ \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix} \begin{vmatrix} A \\ B \\ E \end{vmatrix} - \begin{vmatrix} C \\ D \\ F \end{vmatrix} \begin{vmatrix} A \\ B \\ E \end{vmatrix} \right\} - \begin{vmatrix} A \\ B \\ E \end{vmatrix} \begin{vmatrix} C \\ D \\ F \end{vmatrix}$$

$$= \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ E \end{vmatrix} \begin{vmatrix} C \\ F \end{vmatrix} \begin{vmatrix} D \\ F \end{vmatrix} - 2 \begin{vmatrix} A \\ E \end{vmatrix} \begin{vmatrix} B \\ E \end{vmatrix} \begin{vmatrix} C \\ D \\ F \end{vmatrix} + \begin{vmatrix} A \\ B \\ E \end{vmatrix} \begin{vmatrix} C \\ D \\ F \end{vmatrix}.$$

By the rule for simple factors, the order is, after reduction,

$$\left\{\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\right\}^2.$$

It is evident, indeed, that E and F must both vanish identically.

(4.) The order of the system $\begin{matrix}A.B.C.D=0\\E\qquad\quad=0\end{matrix}$ is

$$\begin{matrix}A\\E\end{matrix}\left|\begin{matrix}B\\E\end{matrix}\right|\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|\overset{*}{=}\begin{matrix}A\\E\end{matrix}\left|\begin{matrix}B\\E\end{matrix}\right|\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|$$

$$-6^{\dagger}\left\{\begin{matrix}A\\B\\E\end{matrix}\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|-\left\{\begin{matrix}A\\B\\E\end{matrix}\left|\begin{matrix}C\\D\\E\end{matrix}\right|-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|\right\}-2\left\{\begin{matrix}A\\B\\C\\E\end{matrix}\left|\begin{matrix}D\\E\end{matrix}\right|-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|\right\}-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|\right\}$$

$$-3\left\{\begin{matrix}A\\B\\E\end{matrix}\left|\begin{matrix}C\\D\\E\end{matrix}\right|-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|\right\}-4\left\{\begin{matrix}A\\B\\C\\E\end{matrix}\left|\begin{matrix}D\\E\end{matrix}\right|-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|\right\}-\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|$$

$$=\begin{matrix}A\\E\end{matrix}\left|\begin{matrix}B\\E\end{matrix}\right|\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|-6\left|\begin{matrix}A\\B\\E\end{matrix}\right|\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|+8\left|\begin{matrix}A\\B\\C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|+3\left|\begin{matrix}A\\B\\E\end{matrix}\right|\left|\begin{matrix}C\\D\\E\end{matrix}\right|-6\left|\begin{matrix}A\\B\\C\\D\\E\end{matrix}\right|.$$

I find that this order reduces to

$$\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left\{\frac{\nu^2}{n^2}-8\frac{\mu\nu}{mn}+12\mu^2+\alpha\,(8\mu-3\nu)+\alpha^2\right\}.$$

The cases which involve four factors in one of the equations belong to the four classes just treated of. But it will be observed that the four conditions necessary to the problem enable us to make a factor in each of the equations vanish identically. We have consequently to consider some other cases. In order to get the terms corresponding to these in the sought result, I proceed as follows.

Since these terms involve the vanishing of a factor in each equation, the common argument must be of the form $\frac{\mu}{m}\left(\frac{\mu}{m}+\alpha\right)\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)$. Moreover, the cases which we have to consider are represented by

A.B.C=0	A.B.C=0	A.B.C=0	A.B=0	A.B=0	A=0
D.E.F=0	D.E =0	D =0	D.E=0	D =0	D=0

† We are enabled to abbreviate by numerical factors, because the orders of $\begin{matrix}A\\C\\E\end{matrix}\left|\begin{matrix}B\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|$, $\begin{matrix}A\\D\\E\end{matrix}\left|\begin{matrix}B\\E\end{matrix}\right|\left|\begin{matrix}C\\E\end{matrix}\right|$, &c., are evidently the same as the order of $\begin{matrix}A\\B\\E\end{matrix}\left|\begin{matrix}C\\E\end{matrix}\right|\left|\begin{matrix}D\\E\end{matrix}\right|$: and so in other cases.

and the additional terms may be written

$$\left\{ mn\frac{(m-1)(m-2)(n-1)(n-2)}{1.2.3.1.2.3}k_1+\Sigma\left(mn\frac{(m-1)(m-2)(n-1)}{1.2.3.1.2}\right)k_2 \right.$$
$$+\Sigma\left(mn\frac{(m-1)(m-2)}{1.2.3}\right)k_3+mn\frac{(m-1)(n-1)}{1.2.1.2}k_4+\Sigma\left(mn\frac{m-1}{1.2}\right)k_5$$
$$\left.+mn\,k_6\right\}\frac{\mu\nu}{mn}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right).$$

Now, for $(m=3, n=3)$, $(m=3, n=2)$, $(m=3, n=1)$, $(m=2, n=2)$, $(m=2, n=1)$, $(m=1, n=1)$, all the terms belonging to the first four classes necessarily vanish. And since the cases now under consideration must be considered as cases of degenerate quartics, and we have for the conditions in order that two quartics ϕ, ψ may be coincident to a constant factor *près*

$$\frac{\phi}{\psi}=k$$

for all values of the variable; it appears that all the cases, in which the number of factors of each equation involved is less than 4, give rise to the order cipher (being, in fact, impossible cases), except when $m=n=1$, and in this instance the order is

$$\frac{\mu\nu}{mn}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right)$$

since the factors identically vanish.

I say that $\frac{\phi}{\psi}=k\,(k=0<\infty)$ cannot be true for all values of the variables, if we make a factor in each function vanish identically; for we have no further equations of conditions at our disposal, having satisfied four in making the factors vanish identically. Consequently, if the ratio is constant, it must be because the remaining factors of the quartics are in fact independent of the variables, *i. e.*, we must have $m=1$, $n=1$.

We are thus enabled to determine k_1, k_2, &c., and the result is, that the terms to be added in consequence of these special cases, are given by

$$\left\{\frac{m(m-1)(m-2)}{1.2.3}3-\frac{m(m-1)}{1.2}2+m\right\}\left\{\frac{n(n-1)(n-2)}{1.2.3}3-\frac{n(n-1)}{1.2}2+n\right\}$$
$$\times\frac{\mu}{m}\frac{\nu}{n}\left(\frac{\mu}{m}+\ \right)\left(\frac{\nu}{n}+\alpha\right),$$

or

$$\frac{m(m-2)(m-3)n(n-2)(n-3)}{4}\frac{\mu\nu}{mn}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right).$$

It remains now to supply, by means of the formula of combinations, the numerical coefficients, and form the symmetrical terms; and our

final result is

$$\frac{mn}{1.2.3.4}\left\{\begin{array}{l}(m-1)(m-2)(m-3)(n-1)(n-2)(n-3)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)^4\\ +\Sigma(m-1)(m-2)(m-3)(n-1)(n-2)6\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)^2\\ +\Sigma(m-1)(m-2)(m-3)(n-1)\\ \quad\times\left\{4\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\nu}{n}-\frac{3\mu}{m}-\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)+3\left[\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\right]\right\}^2\\ +\Sigma(m-1)(m-2)(m-3)\frac{\nu}{n}\left(\frac{\nu}{n}+\alpha\right)\left(\frac{\nu^2}{n^2}-8\frac{\mu\nu}{mn}+12\frac{\mu^2}{m^2}\right)\\ \qquad\qquad+\alpha(8\mu-3\nu)+l\alpha^2\\ +m(m-2)(m-3)n(n-2)(n-3)6\frac{\mu\nu}{mn}\left(\frac{\mu}{m}+\alpha\right)\left(\frac{\nu}{n}+\alpha\right)\end{array}\right\}$$

16. It may be well to remark that the quantics may be separated into factors of various degrees, and in this way the processes may be altered in form though not in principle. This circumstance is, however, more important in reference to quantics containing a greater number of variables. In practice, also, it will be found that a valuable check, to test the correctness of our operations, is furnished by the fact that where $\frac{\lambda^p}{l^p}$ appears in the arguments of the unreduced formula expressing an order, the total coefficient of $\frac{\lambda^p}{l^p}$ must be divisible by lp.

What has been termed by Dr. Salmon the "weight" of the systems corresponding to the orders obtained, may be derived immediately by the operation $\lambda'\frac{d}{d\lambda}+\mu'\frac{d}{d\mu}+\&c.+\beta\frac{d}{d\alpha}$,

λ', m', &c., β being the orders and increment corresponding to those of the secondary variables.

In preparing this paper, I have endeavoured, as far as possible within my limits, to indicate a method of procedure which I believe will be found useful in dealing with problems of the nature here discussed. I have confined myself to binary systems. The same method is applicable to systems of higher quantics separable into linear factors. The results are interesting, because, although obtained subject to this limitation, they are true of quantics generally. I was led to this conclusion from considerations which I cannot now enter on, and actually obtained and applied some formulæ which I afterwards found had been arrived at by Dr. Salmon in a different manner. The discussion of these problems must be reserved for another occasion.

The following communication was made for Mr. Townsend by Prof. Hirst:—

Solution, by the Method of Ordinary Homographic Division, of the Problem, "To inscribe in a given Ruled Quadric a Polygon of any given order whose sides shall pass in any prescribed order of sequence through an arbitrary system of given points in space." By Rev. Richard Townsend, M.A., F.R.S., Fellow of Trinity College, Dublin.

Case 1.—If the number of points be even. Assuming arbitrarily any three generators A, B, C of either (but of the same) system on the surface, draw through them and through the first of the given points P_1 a first triad of planes intersecting the surface again in three first derived generators A_1, B_1, C_1 of the opposite system; through these latter again and through the second of the given points P_2 draw a second triad of planes intersecting the surface again in three second derived generators A_2, B_2, C_2 of the original system; through these latter again and through the third of the given points P_3 draw a third triad of planes intersecting the surface again in three third derived generators A_3, B_3, C_3 of the opposite system; continue the same process until all the points, taken in the prescribed order, are exhausted, which will give three n^{th} derived generators A_n, B_n, C_n of the original system; the two double generators $M = M_n$ and $N = N_n$ of the two homographic systems determined by the three pairs of correspondents A and A_n, B and B_n, C and C_n of the original system will be those of that system on which the first vertices of the required polygons lie. The same process, commenced from any three arbitrarily assumed generators A′, B′, C′ of the opposite system, and continued in the same manner until all the given points taken in the prescribed order are exhausted, will give the two generators $M' = M'_n$ and $N' = N'_n$ of that system on which they also lie. The four vertices P, Q, R, S of the skew quadrilateral formed by the two pairs of double generators M and N, M′ and N′ will consequently be the required vertices themselves; and the problem accordingly admits in this case of *four* solutions, which are evidently all real or all imaginary together.

Case 2.—If the number of points be odd. Assuming arbitrarily, as before, any three generators A, B, C of either (but of the same) system on the surface, derive from them, as before, *two* triads of final generators A_n, B_n, C_n and A_{-n}, B_{-n}, C_{-n} of the opposite system, by application of the same process as before to the given system of points, taken first in the prescribed order P_1, P_2, P_3, and P_n, and then in the opposite order P_n, P_{n-1}, P_{n-2}, and P_1; the two double generators

$M_n = M_{-n}$ and $N_n = N_{-n}$ of the two homographic systems determined by the three pairs of correspondents A_n and A_{-n}, B_n and B_{-n}, C_n and C_{-n} will be those of that system on which the first vertices of the required polygons lie; and their intersections P and Q, each with the one corresponding to itself of the two M and N of the original system, derived from them by reversing the above processes of construction, will be the required vertices themselves; the problem consequently admitting in this case of but *two* solutions, instead of *four* as in the former case.

The reason of this difference in the numbers of solutions in the two cases, appears from the circumstance, which is evident from the nature of each, that, of the two homographic systems of points in space determined by the first and last extremities of all open polygons inscribed in the surface, whose sides pass in the prescribed order through the systems of given points, the four generators M and N, M′ and N′ in the former case are all double lines; while in the latter case, on the contrary, no double line exists on the surface. In fact, of the tetrahedron determined by the four intersections P, Q, R, S of the two pairs of determining generators in the two cases, the four vertices are all double points, the four faces are all double planes, and the six edges are all double lines, in the former case; while, on the contrary, but two of the vertices P and Q are double points, but two of the faces RPS and RQS are double planes, and but two of the edges PQ and RS are double lines, in the latter case.

As regards the remaining elements of the tetrahedron in the latter case; the two pairs of corresponding lines M and M_n, N and N_n being such, in relation to the two homographic systems of points, that every point on either has its correspondent on the other, to whichever system it be regarded as belonging, each of those four lines is consequently interchangeable with its correspondent in the two systems; and the two pairs of edges PR and PS, QR and QS being interchangeable lines, the two vertices R and S are interchangeable points, and the two faces PRQ and PSQ are interchangeable planes, in the systems; and therefore, by the fundamental property of homographic systems in involution, all pairs of corresponding points on the double line RS, and all pairs of corresponding planes through the double line PQ are interchangeable in the systems.

In Case 1, the solution of the problem may be either partially or wholly indeterminate. The former, if the three pairs of correspondents in either triad of pairs of corresponding generators A and A_n, B and B_n, C and C_n, or A′ and A'_n, B′ and B'_n, C′ and C'_n coincide, those in the other triad not coinciding; and the latter, if the three pairs in both triads coincide; every generator of one system in the former case, and of both systems in the latter case, being then a double line of the two homographic systems in question. In the former case,

every point on either double generator of the system for which the three pairs do not coincide, and in the latter case every point on the entire surface, is a first vertex of a polygon satisfying the conditions of the problem. In the former case, the two double generators on which all the first vertices lie may coincide, in which case the first vertices are of course limited to the one line of coincidence.

In Case 2, the solution of the problem may be partially, but can never be wholly, indeterminate. If, of the triad of pairs of corresponding generators A_n and A_{-n}, B_n and B_{-n}, C_n and C_{-n}, the three pairs of correspondents coincide, it will be partially indeterminate; every point on the section of the surface by the plane passing through the three points of intersection of A with A_n or A_{-n}, of B with B_n or B_{-n}, and of C with C_n or C_{-n}, being then a double point of the two homographic systems in question; which is evident from the considerations (1) that the two corresponding generators A and A_n or A_{-n} by their simultaneous variation divide every plane section of the surface homographically, and (2) that their two divisions of the particular section in question have three double points. That, in the same case, the several closing sides of all the open polygons, whose first vertices do not lie on that section, pass through the vertex of the cone enveloping the surface along that section, is also evident from the consideration, that, as every tangent plane to that cone is a double plane, therefore every line passing through their common point of intersection is a double line, of the two homographic systems determined by the extremities of those sides.

Note.—The preceding results, so far as they extend, are in entire accordance with those obtained by the late Sir W. R. Hamilton, Astronomer-Royal of Ireland, who, in the year 1849, applied his "Calculus of Quaternions," then almost in its infancy, to the solution of the above problem in all its generality for convex as well as for ruled quadrics, and who was, if I mistake not, the first geometer who successfully considered the question at all. In the month of June, 1849, he made a preliminary communication to the Royal Irish Academy, of which he was ex-President, stating without demonstration the results he had up to that time arrived at, and expressing at the same time his intention to submit his demonstrations *in extenso* to the Committee of Reference for publication in the Transactions of that Body; an intention however which, engaged as he was for the remainder of his life in perfecting the instrument of investigation itself, he never afterwards carried out. In this communication he pointed out clearly the distinction between the two cases of n odd and n even, and stated that the problem admitted, in general, in the former case of but *two* solutions, real or imaginary according to circumstances,

given by the two intersections with the surface of a single determinable line; and in the latter case, on the contrary, of *four* solutions, given by the four intersections with the surface of a pair of reciprocal lines with respect to itself, and consequently all four real or imaginary together for ruled, and two real and two imaginary for convex surfaces. And in the month of May, 1850, he made a further communication to the same Body, showing, by a series of simple geometrical transformations, very analogous in their first steps to those employed by Poncelet in his well-known *indirect* solution of the corresponding problem for plane conics, how to determine the one line or the two lines of solution in the two cases respectively; and thus *completely* solving the problem in all its generality, which, so far as I am aware, has never been done before or since by any other geometer.* Accounts of these two interesting communications will be found in the "Proceedings of the Royal Irish Academy" for the months in which they were respectively made.

The following communication was accidentally omitted in the report of the Meeting of Nov. 22, 1866:—"On Analogues in the Parallelopiped to Properties of the Parallelogram." By J. J. Walker, M.A. The author stated and proved the following theorems:—

1. The tetrahedra having a common vertex, and standing on the halves of two faces of the parallelopiped which have a common edge as bases, are together equal to the tetrahedron having the same vertex, and standing on half the diagonal parallelogram of which that edge is also a side.

2. The volumes of the three tetrahedra (taken with appropriate signs) having any point in space as a common vertex, and which stand on the halves of any three cointersecting faces of a parallelopiped as bases, are together equal to the volume of the tetrahedron having the same vertex and having as base the triangle whose corners are the points of co-intersection of those faces and the corners opposite to t of any two of the three faces. The signs would be all positive if the common vertex of the four tetrahedra were situate within the tetrahedral angle formed by producing the three faces beyond their point of co-intersection.

* A very elegant construction, at once simple, direct, and general, has recently been given for the case of n odd, by Mr. M. Gardiner, C.E., late Scholar of Queen's College, Galway, Ireland, and since Professor of Mathematics in St. John's College, Sydney, New South Wales. See *Quarterly Journal of Pure and Applied Mathematics*, Vol. VII., pp. 284—301.

February 28*th*, 1867.

Prof. SYLVESTER, President, in the Chair.

The Rev. Hugh Martin, M.A., was elected a Member of the Society.

The following communications were read:—

"Extensions of Möbius and Feuerbach's theorems relating to a variable and four fixed points, with applications to the proof of properties of focal curves:" by M. W. Crofton, B.A.

"On Properties of Homographic Systems of Points on a Ruled Surface of the Second Order:" by the Rev. R. Townsend, F.R.S.

"On the Law of Universal derivation of point from point on Cubic curves:" by Prof. Sylvester, F.R.S.

March 28*th*, 1867.

Prof. SYLVESTER, President, in the Chair.

Mr. John Whitt, B.A., was elected a Member of the Society.

Professor Cayley mentioned a theorem included in Prof. Sylvester's theory of derivation of the points of a cubic curve. Writing down the series of numbers 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, &c., viz., all the numbers not divisible by 3, then (repetitions of the same number being permissible) taking any two numbers of the series, we have in the series a third number, which is the sum or else the difference of the two numbers (for example, 2, 2 give their sum 4, but 2, 7 give their difference 5), and we have thus a series of triads, in each of which one number is the sum of the other two. The theorem is, that it is possible on a cubic curve to construct a series of points, such that, denoting them by the above numbers respectively, then for any triad of numbers as aforesaid the points denoted by the three numbers respectively lie *in lineâ*. And the theorem gives its own construction: in fact, the series of triads is 112, 224, 145, 257, 178, 248, &c. Take 1, an arbitrary point on the cubic, then (by the theorem) the triad 112 shows that 2 is the tangential of 1; 224 shows that 4 is the tangential of 2; 145 that 5 is the third point of 1 and 4; 257 that 7 is the third point of 2 and 5. So far we have no theorem; we have merely, starting from the point 1, constructed by an arbitrary process the points 2, 4, 5, and 7. But going a step further; 178 and 448 show, the first of them, that 8 is the third point of 1 and 7, the second

of them, that 8 is the tangential of 4. We have here the theorem that the third point of 1 and 7 is also the tangential of 4. Similarly, 10, 11, 13 are each of them (like 8) determined by two constructions; 14, 16, 17, 19, each of them by three constructions, and so on: the number of constructions increasing by unity for each group of four numbers. And the theorem is, that these constructions, 2, 3 or more, as the case may be, give always one and the same point. Prof. Cayley mentioned that on a large figure of a cubic curve he had, in accordance with the theorem, constructed the series of points 1, 2, 4, 5, 7, 8, 10, 11, 13, 14.

Prof. Sylvester made a communication "On the Theory of Residuals."

Mr. Walker gave a proof of the Rectangle of Forces; and demonstration of the principle of the Parallelogram of Forces. These are new and interesting specimens of a class of proofs on the character of which mathematical opinion has been much divided. The interest of these proofs now turns on the question, how much and what there is of a physical character in the assumptions on which they are founded: in fact, what there is to prevent their being purely geometrical, and having a conclusion true of mathematical necessity.

April 25th, 1867.

Prof. SYLVESTER, President, in the Chair.

M. Chasles was elected a Foreign Member of the Society.

Prof. Clerk-Maxwell, F.R.S., Dr. Robertson, F.R.S.E., and Rev. Thos. Smith, M.A., were elected Members.

The following paper was read:—

On the Conic Octagram.

By A. De Morgan, Esq., Vice-President.

The most general theorem I require is the particular case I now state. If of the n^2 intersections of two n-ic curves $n(n-2)-\frac{1}{2}(n-3)(n-4)$ lie on an $(n-2)$ic, so also do $\frac{1}{2}(n-3)(n-4)$ more of the intersections, which call *adjuncts*: and the remaining $2n$ intersections lie on a conic. When $n=3$, and the cubics are trilinear, we have Pascal's *hexagram*: we may retain this word as denoting, not a *hexagon*, but a method of drawing or completing a hexagon. Each value of n, if the n-ics be multilinear, gives a direct and inverse $2n$-gram: direct, when by an $(n-2)$ic taken at pleasure we determine $2n$ points of some conic of which nothing is given beforehand; inverse, when, having given five determining points of a conic, we

determine $2n-5$ points more by an inverse construction which determines a convenient $(n-2)$ic.

When $n=3$ or $n=4$, there are no adjunct intersections; and the inverse $2n$-gram is the plainest inversion of a simple construction. But when $n=5$ or upwards, the inverse $2n$-gram, though practicable, requires the introduction of an entirely new element. When $n=5$, for example, and there is one adjunct, the direct construction makes 14 intersections put the 15th on its proper place; but, in the inverse construction, the cubic cannot be discovered without finding it so as to satisfy the condition of the 15th intersection. Consequently, there are no other theorems of the family of Pascal's hexagram except only the octagram. I shall state the methods for: 1. the *hexagram;* 2. the *double trigram*, which is another form of the hexagram; 3. the *octagram;* 4. the *double tetragram*, which is *not* another form of the octagram; 5. the direct form of the *decagram.*

I signify lines by 1, 2, 3, &c.; points of intersection by 12, 23, &c.; a line drawn through two points by 12 | 34, &c. And when A, B are points on a line 3, A3B signifies as much of 3 as is between A and B; and so on.

1. *Hexagram.* Direct: if the hexagon* 142536 have its alternate points (14, 25, 36) collinear, then 123456 is in a conic; and the converse. Inverse: if 123456 be a conic hexagon, of which are known 61, 12, 23, 34, 45, and the direction of 5, then (14 | 25 . 3) | 61 meets 5 in 56, the sixth point. After the data are laid down, two new lines determine one point.

2. *Double trigram.* This is a modification of the hexagram which depends on Desargues's *coaxal* triangles. These triangles form a conic hexagon; that is, if 123, 456 be such triangles, 12, 23, 34, 45, 56, 61 are† co-conic. The inverse theorem is only worth mention for analogy: the construction is more complex than the hexagram. Let 123 be an inscribed triangle, and 456 another, of which 45, 46, and the direction of 5, are given. From the intersection of 12 | 45 and 13 | 46 draw a line 7 to the intersection of 23 | 46 and 5. Then 72 | 13 meets 5 in the required point 56. The coaxal triangles are 45 (23 | 46) and 12 (72 | 13). And 7 is the Pascalian line of the hexagon 13, 46, 32, 21, 45, 56.

3. *Octagram.* Direct: choose a conic (usually a pair of straight

* This presentation of the second hexagon was actually suggested to me by observing that *Blaise Pascal* has two hexagrams, and the jocose inference that there ought to be two hexagons in the theorem. My own names are both octagrams; but, though I bow before the coincidence, I have no suggestion to acknowledge.

† Desargues was the teacher of Pascal, who expresses his obligation in terms of unusual strength. It is certain, by the date of printed satire, that the theorem called *la Pascale* was known by that name before Pascal was twenty years old. I feel sure that Pascal's active genius was led by the coaxal triangles to the hexagram.

lines) as *guide*, and inscribe an octagon of which the successive chords are numbered by procession by 5, rejecting 8; as, 16305274. Then 01, 12, 23, 34, 45, 56, 67, 70 are co-conic; or, 01234567 is an octagon inscribed in a conic. Inverse: in the conic octagon 01234567 are given 1, 2, 3, 4, and the directions of 0 and 5: required 56, 67, 70. Find 14, 50, 25, 30, 14 | 50, 25 | 30, 14 | 50 . 2, 14 | 50 . 3, 25 | 30 . 1, 25 | 30 . 4. Let 6 and 7 be

14 | 50 . 3 | 25 | 30 . 1 and 14 | 50 . 2 | 25 | 30 . 4.

Then 56, 67, 70 will be on the conic, and 01234567 is an inscribed octagon. The *guide* is the system 14 | 50, 25 | 30; and three points are determined by four lines over and above data. By varying the directions of 0 and 5, and drawing the corresponding parts of all the constructions together, the laying down of a large number of points is very easy and rapid work.

4. *Double tetragram.* When I explained the octagram to the Society, Mr. Cotterill suggested a known theorem which seemed likely to be the direct octagram; but, on looking into it, I found it to be the direct double tetragram, which it suggested in another form. In a conic (usually two straight lines) inscribe two tetragrams 1630 and 5274; then 1234. 5670 are tetragons in another conic. Inversely: let the tetragon 1234 be given; and in 5670 the point 50 and the directions of 5 and 0. Find 51, 53, 02, 04, 51 | 02, 53 | 04; these lines are the guide. Let 6 and 7 be

53 | 04 . 2 | 51 | 02 . 4 and 53 | 04 . 1 | 51 | 02 . 3.

Then 56, 67, 70 will be on the conic, and 5670 is a second tetragon.

Owing to our better command of language for a tetragon than for a hexagon, we may describe this last construction as follows:—Let one pair of opposite sides be *bases*, and the other pair *sides*; points in their lines being base-points and side-points. Given a tetragon, and a fifth point of the conic in which two lines meet, take the base-points in which the fifth line, and the side-points in which the sixth line cuts the tetragon. Join these two and two, base-point to side-point; in each of the joining lines there is a new base-point and a new side-point. Join these last, base-point to base-point, and side-point to side-point; the two lines so drawn complete the second tetragon.

5. *Direct decagram.* Take any three straight lines P, p, ϖ; any curve of the third order would do as well. Capital letters are points on P; small letters on p. Describe the decagon

A0a1B2b3C4c5D6d7E8e9A

in the following way. Take A0a, a1B, B2b, b3C, C4c at pleasure, and let the points in which 0, 1, 2, 3, 4, thus laid down, meet ϖ be 05, 16, 27, 38, 49. Then draw c5D through c and 05, meeting P in D, &c., as from the above description. It will at last be found that 9, as obtained from e and 49, passes through A; or the adjunct fails as

required by the theorem. This done, 03, 36, 69, 92, 25, 58, 81, 14, 47, 70, are co-conic; or 0369258147 is a conic decagon. Incidentally, we have solved the question of laying down fifteen points that shall lie five and five on three lines, and in two ways three and three on five lines. The direct dodecagram would be, I suppose, as easy, and would give twenty-four points lying six and six on four lines, and four and four on six lines in two ways, and so on.

I obtain what precedes from accidentally coming upon two of the new points of the octagram in the following way; and the third soon followed. If a quartic pass through nine *poristic** points of a cubic, the three remaining intersections are collinear. This is demonstrated as follows; and all the cases of the general theorem in which there are no adjuncts can be treated in the same way. Let the poristic points be the intersections of two cubics V_3, W_3, and let A_1, B_1 be straight lines, any whatever. Then $A_1V_3+B_1W_3$ has five disposable ratios of constants, just enough to make it any quartic which can pass through the nine poristics $(9+5=14)$. This intersects V_3 in B_1W_3, that is, the nine poristics in W_3, and the remaining three intersections, which must be in B_1.

Prof. Cayley then communicated a theorem relating to the locus of the ninth of the points of intersection of two cubics, seven of these points being fixed, while the eighth moves on a straight line.

The following paper was read:—

Proof of an Arithmetical Theorem leading, by means of Gauss's Fourth Demonstration of Legendre's Law of Reciprocity, to the extension of that Law. By Morgan Jenkins, B.A., late Scholar of Christ's College, Cambridge.

It is a well-known theorem, that if all the odd numbers less than any given odd number N be multiplied by any number K, prime to N, and the residues less than N, after division by the modulus N, be formed, turning every even residue into the corresponding negative odd residue, that the latter series of residues will be numerically the same as the former.

Then I propose to show that the number of negative terms in the latter series is even or odd according as $\left(\frac{K}{N}\right)=+1$, or $=-1$; where $\left(\frac{K}{N}\right)$ receives Jacobi's extended signification.

* I call these points poristic because they are poristic: they are so situated that the usually definite problem of drawing a cubic through them becomes a porism. Not *porismatic;* this is no better than *sophismatic*, instead of sophistic or sophistical. Certainly *αἰνιγματικός* is in the lexicon; but Liddell and Scott decline to produce any authority for it, and only remark that it is = *αἰνικτήριος*.

Let $O[1, X]$ denote the series of odd numbers less than any number X;

$O_x[1, X]$ a group of the series such that every number in the group has x for the G.C.D. of itself and X. (This does not, of course, necessarily imply that x is the G.C.D. of all the numbers in the group.)

Hence $O_1[1, X]$ would signify a group of all the odd numbers less than and prime to X;

also, let $O[K, X]$, $O_x[K, X]$ denote the series and a group derived from $O[1, X]$, $O_x[1, X]$ respectively in the manner above described, K being prime to X;

also, let $P[1, X]$, $P[K, X]$ ... &c., denote the corresponding products of the series and groups.

and $\phi(X)$ the number of numbers less than and prime to any number X;

also, let the odd number $N = a^\alpha b^\beta c^\gamma \ldots$, where $a, b, c \ldots$ are odd prime factors.

$u = a^p b^q c^r \ldots$ be any divisor of N except 1; so that $p, q, r \ldots$ take all integral values between 0 and α, 0 and $\beta \ldots$ respectively, inclusive; except that they are not all equal to 0.

Hence $u' = \dfrac{N}{u}$ may be made equal to any divisor of N except N, but including 1.

Then $O[1, N]$ may obviously be divided into groups of the form $O_1[1, N], \ldots O_{u'}[1, N]$. Also, if Mu' be any number in $O_{u'}[1, N]$, since u' is G.C.D. of Mu' and N or uu', M is prime to u; and, since Mu' is $< uu'$ and odd, M is $< u$ and odd. Conversely, if M be any odd number $<$ and prime to u, Mu' would be a number in $O_{u'}[1, N]$. Hence $O_{u'}[1, N]$ may be obtained by multiplying every number in $O_1[1, u]$ by u'.

Again, the $\phi(u)$ numbers $<$ and prime to u are equally distributed between odd and even numbers. Therefore there are $\frac{1}{2}\phi(u)$ numbers in $O_1[1, u]$ and $O_{u'}[1, N]$. Hence, by taking a half system of residues, viz. all odd numbers $<$ and prime to u, we could show, since K is prime to N, and therefore to u, that $K^{\frac{1}{2}\phi(u)} \equiv \pm 1 \pmod{u}$:—a known result which is an immediate deduction from Euler's generalization of Fermat's theorem, *i.e.* $K^{\phi(u)} \equiv 1 \pmod{u}$: also that

$$P_1[K, u] = \{K^{\frac{1}{2}\phi(u)} \pmod{u}\}\, P_1[1, u],$$

where $K^{\frac{1}{2}\phi(u)} \pmod{u}$ is supposed to have its least numerical value of ± 1.

But if r_p, r_q be two of the numbers in $O_1[1, u]$ such that

$$Kr_p \equiv \pm r_q \pmod{u}, \text{ then } Kr_p u' \equiv \pm r_q u \pmod{N \text{ or } uu'}.$$

Hence the absolute magnitude of the numbers in every group $O_{u'}[K, N]$ is the same as that of the numbers in the corresponding group $O_{u'}[1, N]$, and it is obvious that

$$P_{u'}[K, N] = \{K^{\frac{1}{2}\phi(u)} \pmod{u}\}\, P_{u'}[1, N] \quad \ldots\ldots\ldots\ldots(1);$$

and therefore $P[K, N] = [\Pi\{K^{\frac{1}{2}\phi(u)} \pmod{u}\}]\, P[1, N]$ .....(2),

where u' takes all possible values, and Π denotes the continued product.

Now $K^{\frac{1}{2}(a-1)} \equiv \pm 1 \equiv \left(\frac{K}{a}\right) \pmod{a}$ by definition of symbol;

$$\therefore K^{\frac{1}{2}(a-1)a^{p-1}} \equiv \left(\frac{K}{a}\right)^{a^{p-1}} \pmod{a^p},$$

or $$K^{\frac{1}{2}\phi(a^p)} \equiv \left(\frac{K}{a}\right) \pmod{a^p}, \text{ since } a^{p-1} \text{ is odd} \ldots\ldots\ldots\ldots\ldots(3);$$

$$\therefore K^{\frac{1}{2}(a-1)a^{p-1}(b-1)b^{q-1}\ldots} \equiv \left(\frac{K}{a}\right)^{(b-1)b^{q-1}\ldots} \pmod{a^p},$$

$$\equiv +1 \pmod{a^p}, \text{ since } (b-1) \text{ is even.}$$

In like manner, $$\equiv +1 \pmod{b^q}, \text{ \&c.}$$

Hence, if u contain more than one prime factor,

$$K^{\frac{1}{2}\phi(u)} \equiv +1 \pmod{u} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4);$$

$\therefore$ from $\Pi\{K^{\frac{1}{2}\phi(u)} \pmod{u}\}$ all values of u may be rejected which contain more than one prime factor; and we need only consider the sign of $\Pi\left[\{K^{\frac{1}{2}\phi(a^p)} \pmod{a^p}\}\{K^{\frac{1}{2}\phi(b^q)} \pmod{b^q}\}\ldots\right]$.

But, by (3), every term $K^{\frac{1}{2}\phi(a^p)} \equiv \left(\frac{K}{a}\right) \pmod{a^p}$,

and every term $$K^{\frac{1}{2}\phi(b^q)} \equiv \left(\frac{K}{b}\right) \pmod{b^q} \ldots \text{ \&c.};$$

and the limits of p, q ... for powers of single prime factors are 1 to α, 1 to β ... respectively, inclusive,

$$\therefore \Pi\{K^{\frac{1}{2}\phi(u)} \pmod{u}\} = \left(\frac{K}{a}\right)^{\alpha}\left(\frac{K}{b}\right)^{\beta}\ldots$$

$$= \left(\frac{K}{N}\right) \text{ by definition of symbol;}$$

$$\therefore \text{by (2), } P[K, N] = \left(\frac{K}{N}\right) P[1, N].$$

Application of the theorem:—Using the account of Gauss's fourth demonstration of Legendre's Law of Reciprocity ("Summatio Serierum," &c., Comm. recent., Soc. Gött., Vol. I., Aug. 24), given, with

some additions to the notation, by Prof. H. J. S. Smith, in his Report on the Theory of Numbers (Part I.), (Rep. of Brit. Ass. 1859, Vol. xxix., p. 246), and italicizing the alterations which are necessary for the extension of the demonstration; then, if $i = \sqrt{(-1)}$, N be any uneven number, $\sqrt{N}$ its positive square root, $r = \cos\frac{2\varpi}{N} + i\sin\frac{2\varpi}{N}$, and $1+r^{K}+r^{4K}+r^{9K}+\ldots+r^{(N-1)^2K}$ be denoted by $\psi[K, N]$, we have

$$\psi[1, N] = i^{\frac{1}{4}(N-1)^2}\sqrt{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

This equation may be deduced from the following one, (B), proved by Gauss in the "Summatio Serierum," &c., viz.

$$1+\rho+\rho^4+\rho^9+\ldots+\rho^{(N-1)^2} = (\rho-\rho^{-1})(\rho^3-\rho^{-3})\ldots(\rho^{(N-2)}-\rho^{-(N-2)})\ldots(B),$$

if N be any uneven number, and ρ any primitive root of $x^N-1=0$.

If we make $\rho = r$, and write θ for $\frac{2\varpi}{N}$, we have

$$\psi[1, N] = (2i)^{\frac{N-1}{2}}\sin\theta\sin 3\theta\ldots\sin(N-2)\theta\ldots\ldots\ldots(B'),$$

from which (A) follows.

If we make $\rho = r^K$, *where* K *is any number prime to* N, we have

$$\psi[K, N] = (2i)^{\frac{N-1}{2}}\sin K\theta\sin 3K\theta\ldots\sin(N-2)K\theta\ldots\ldots(C');$$

and therefore, by means of (B'), (C'), *and the arithmetical theorem of this paper,*

$$\psi[K, N] = \left(\frac{K}{N}\right)\psi[1, N]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(D),$$

where the symbol $\left(\frac{K}{N}\right)$ has Jacobi's extended signification.

Also, by actual multiplication, *since* K *and* N *are prime to each other,*

$$\psi[K, N]\times\psi[N, K] = \psi[1, KN]\ldots\ldots\ldots\ldots\ldots(E).$$

Hence, from (D), $\left(\frac{K}{N}\right)\left(\frac{N}{K}\right) = \frac{\psi[1, KN]}{\psi[1, K]\psi[1, N]}$ $\ldots\ldots\ldots\ldots(F).$

If we make K uneven, *in order to give significance to* $\left(\frac{N}{K}\right)$, and substitute for the functions ψ their values given by the equation (A), we find an equation which gives a relation equivalent to that assigned in the extended law of reciprocity; viz., that, *if* K *and* N *be any two uneven numbers prime to each other,* $\left(\frac{K}{N}\right) = \left(\frac{N}{K}\right)$, unless K and N are both of the form $4m+3$.

May 23rd, 1867.

Prof. SYLVESTER, President, in the Chair.

The following Paper was read:—

*On Various Properties of Bicircular Quartics.**

By M. W. CROFTON, B.A. Trin. Coll. Dub., Mathematical Master in the Royal Military Academy, Woolwich.

1. It is known that, if 1, 2, 3, 4 be any four points forming a convex quadrilateral; ρ_1, ρ_2, ρ_3, ρ_4 their distances from any point R in the plane; and if α, β, γ, δ denote the triangles (234), (134), (124), (123),

$$\alpha\rho_1^2-\beta\rho_2^2+\gamma\rho_3^2-\delta\rho_4^2=C \quad \ldots\ldots(1),$$

where the constant

$$C = (\text{area of quadrilateral } 1234) \,.\, (IX \,.\, X3-2X \,.\, X4);$$

X being the intersection of the diagonals.

Hence, if 1, 2, 3, 4 are concyclic,

$$\alpha\rho_1^2-\beta\rho_2^2+\gamma\rho_3^2-\delta\rho_4^2=0 \quad \ldots\ldots(2).$$

I do not know if it has been noticed, that Möbius' relation between the distances of any four points on a conic from a focus follows immediately from (1): thus, let R, S be the foci of an hyperbola passing through any four points 1, 2, 3, 4, whose equation is

$$\rho=\sigma+a;$$

we shall have, from (1),

$$\alpha(\sigma_1+a)^2-\beta(\sigma_2+a)^2+\gamma(\sigma_3+a)^2-\delta(\sigma_4+a)^2=C.$$

But, by (1), $\alpha\sigma_1^2-\beta\sigma_2^2+\gamma\sigma_3^2-\delta\sigma_4^2=C$; and $\alpha-\beta+\gamma-\delta=0$;

$$\therefore \alpha\sigma_1-\beta\sigma_2+\gamma\sigma_3-\delta\sigma_4=0 \quad \ldots\ldots(3);$$

a relation connecting the distances of a focus of a conic from any four points on the curve:† the same proof obviously applying to the ellipse.

Equation (3) may be considered as the quadri-polar equation of the locus of foci of all conics through 1, 2, 3, 4: it may be shown to be a sextic curve. When 1, 2, 3, 4 are concyclic, it may be proved (see Mr. Burnside's solution in *Educational Times*, December, 1866; see also Art. 7) that (3) represents the two circular cubics whose foci are 1, 2, 3, 4. It is important to observe, however, that (3), without any change of sign, only represents the serpentine branches of those

* This Paper includes the communication made by Mr. Crofton at the meeting held on February 28th, 1867.

† As the two branches of an hyperbola are distinct conics in the vectorial system, the above does not apply to the cases where 1, 2, 3, 4 are not all on one branch, and where therefore the quadrilateral may be re-entrant. It is, however, easy to show, as above, that (3) is still true in those cases, by certain changes of signs. Also, with changes of sign, (1) holds for a re-entrant quadrilateral.

cubics. Hence the locus of the foci of all conics* through four concyclic points is the *serpentine branches* of the two circular cubics whose foci are those points.

2. Let R, S be the foci of a Cartesian oval passing through the concyclic points 1, 2, 3, 4; let its equation be

$$\rho = m\sigma + a;$$

we have, by (2),

$$\alpha(m\sigma_1+a)^2-\beta(m\sigma_2+a)^2+\gamma(m\sigma_3+a)^2-\delta(m\sigma_4+a)^2=0;$$

from which we at once deduce, as before, the equation (3),

$$\alpha\sigma_1-\beta\sigma_2+\gamma\sigma_3-\delta\sigma_4=0:$$

so that we have the remarkable theorem due to Prof. Sylvester (see *Educational Times*, April, 1867):—The locus of the foci of *all Cartesians through four concyclic points* is the two serpentine branches of the circular cubics above: as for the conics.

The three foci of any one of the Cartesians lie on one of these serpentine branches. (See a proof by me in *Educational Times*, October, 1866.)

A curious case of this theorem is when 1, 2, 3, 4 are on a straight line. Here it seems to fail for the conics, which reduce to the straight line itself: however, the straight line, considered as an hyperbola, has every point in the plane for a focus, and therefore every point on the cubics. But any number of Cartesians (of the *indented* form) may pass through four points in directum; and the locus of their foci will be the two serpentines whose foci are 1, 2, 3, 4. I am not aware of the variety of circular cubic whose foci are collinear having been examined. If three of the four coincide, it gives the Cissoid.

3. Before proceeding to apply the same method to the foci of the bicircular quartics through four concyclic points, it is necessary to state certain theorems.

If l, m, n are three positive numbers, the complete bicircular quartic in rectangular coordinates includes all the signs in

$$\pm l\rho \pm m\sigma \pm n\tau = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Any one of these combinations, such as $l\rho+m\sigma-n\tau=0$, represents, if it be real, a single closed curve or oval in tripolar coordinates:† the complete quartic (4) consisting of two real ovals only, as we shall see further on. For the sake of clearness, I shall call any real oval, as

* When speaking of a conic and a Cartesian oval, I mean, for clearness, single branches or ovals, viewing the conjugates as different curves. However, the theorems may easily be extended. Thus, if two of the points 1, 2, 3, 4 are on one branch of an hyperbola, and the rest on the other branch, the locus of its foci will be the oval branches of the same two circular cubics.

† The proofs of this and some other propositions I suppress, partly from want of space, partly because I am convinced that they admit of simpler and more elegant proofs than I have succeeded in finding.

$l\rho+m\sigma-n\tau=0$, *a bicircular oval;* the complete system (4), of two conjugate ovals, constituting *a bicircular quartic.*

4. Given the three foci R, S, T, one bicircular oval may be made to pass through any two points $(\rho_1, \sigma_1, \tau_1)$, $(\rho_2, \sigma_2, \tau_2)$. Its equation is

$$\begin{vmatrix} \rho, & \sigma, & \tau \\ \rho_1, & \sigma_1, & \tau_1 \\ \rho_2, & \sigma_2 & \tau_2 \end{vmatrix} = 0, \text{ or}$$

$$\left(\frac{\rho}{\tau}-\frac{\rho_1}{\tau_1}\right)\left(\frac{\sigma_2}{\tau_2}-\frac{\sigma_1}{\tau_1}\right)=\left(\frac{\sigma}{\tau}-\frac{\sigma_1}{\tau_1}\right)\left(\frac{\rho_2}{\tau_2}-\frac{\rho_1}{\tau_1}\right) \quad \ldots\ldots\ldots\ldots(5).$$

This becomes indeterminate if $\rho_1:\sigma_1:\tau_1=\rho_2:\sigma_2:\tau_2$. If this relation hold, the point $(\rho_2\sigma_2\tau_2)$ is inverse to $(\rho_1\sigma_1\tau_1)$ with regard to the circle of the foci, or *primary circle*, as it may be called. Hence, given three foci, all ovals through any given point also pass through the inverse of that point. Also, *every bicircular oval is its own inverse with regard to the primary circle*, or *anallagmatic*, as M. Moutard has termed it. Hence, it is easy to see that, if tangents are drawn at two inverse points, and produced to meet, they are equal in length; and therefore *any two inverse points on a bicircular oval are the points of contact of a double contact circle.* When the two inverse points coincide, the double contact circle becomes the osculating circle, the contact being 4-pointic. Hence a bicircular oval meets the primary circle at right angles in two points, which are points of maximum or minimum curvature.

It is evident that any curve given by a homogeneous linear relation of the distances from any number of fixed concyclic poles

$$\pm l\rho\pm m\sigma\pm n\tau\pm p\upsilon\pm \&c. = 0$$

is also anallagmatic; for, $(\rho\sigma\tau\upsilon\ldots)$ being any point on the curve, its inverse will be $(k\rho, k\sigma, k\tau, k\upsilon\ldots)$; and these satisfy the equation. (See Art. 18.)

5. Given three points 1, 2, 3 on a bicircular oval, and two foci R, S, to find the locus of its other foci.

Let T be one of its remaining foci, its equation being $\tau=\alpha\rho+\beta\sigma$: we shall have

$$\tau_1=\alpha\rho_1+\beta\sigma_1,$$
$$\tau_2=\alpha\rho_2+\beta\sigma_2,$$
$$\tau_3=\alpha\rho_3+\beta\sigma_3.$$

Eliminating α, β, we find

$$\left(\frac{\tau_3}{\rho_3}-\frac{\tau_1}{\rho_1}\right)\left(\frac{\sigma_2}{\rho_2}-\frac{\sigma_1}{\rho_1}\right)=\left(\frac{\tau_2}{\rho_2}-\frac{\tau_1}{\rho_1}\right)\left(\frac{\sigma_3}{\rho_3}-\frac{\sigma_1}{\rho_1}\right).$$

In this equation all the quantities are known constants, except τ_1, τ_2, τ_3, which are the current tripolar coordinates of the focus T, referred to 1, 2, 3 as poles. Hence the required locus is *the bicircular oval passing through* R, S, *whose foci are* 1, 2, 3.

It will hence be easily seen that:—

The four points in which a bicircular oval is cut by any circle will be the foci of another oval passing through the four foci of the first. (This has been proposed by Mr. Clifford in the *Educational Times*, Sept., 1866.)

If a bicircular oval have two given foci, and pass through three fixed points, it will pass through a fourth fixed point—viz., the fourth focus of the bicircular oval passing through the two given foci, and having the three fixed points as foci.

If any bicircular oval pass through three fixed points, its four foci lie on some one bicircular oval, having those three points as foci.

6. If a bicircular oval pass through four fixed concyclic points 1, 2, 3, 4, and have one focus (R) fixed, *the locus of its three other foci consists of the two bicircular ovals which pass through* R, *and have* 1, 2, 3, 4 *as foci.*

This theorem offers the same paradox as that of Mr. Sylvester (Art. 2): for one focus and four points on the curve give but 6 conditions, whereas 8 conditions are required to determine a bicircular oval: so that we might apparently take any point whatever as a second focus.

To prove it, let S, T be two of the other foci, and let

$$\tau = m\rho + n\sigma$$

be any oval passing through 1, 2, 3, 4. Proceeding exactly as in Art. 2, we find

$$\alpha\rho_1\sigma_1 - \beta\rho_2\sigma_2 + \gamma\rho_3\sigma_3 - \delta\rho_4\sigma_4 = 0,$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are the given vectors of the focus R: and this proves the theorem, as we shall see in Art. 7.

From the last theorem in Art. 5 we see that, though the locus consists of two curves, the three remaining foci of any oval through 1, 2, 3, 4 lie on one of those curves. The whole system of curves divide into two families, each having its foci on a distinct curve, as in the conics and Cartesians through four concyclic points.

If the given focus R be the centre of the circle 1234, the locus of the three other foci consists of the same two serpentine circular cubics as have been found for the conics and Cartesians.

Also, if R be any point on one of these circular cubics, one constituent of the locus becomes that circular cubic; which therefore is still the locus of foci for one family of bicircular ovals through 1, 2, 3, 4.

7. To prove that, if $(\rho_1\rho_2\rho_3\rho_4)$ be a given point R, the equation

$$\alpha\rho_1\sigma_1 - \beta\rho_2\sigma_2 + \gamma\rho_3\sigma_3 - \delta\rho_4\sigma_4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

represents the two bicircular ovals passing through R, *whose foci are* 1, 2, 3, 4.

We have

$$\alpha\rho_1\sigma_1 + \gamma\rho_3\sigma_3 = \beta\rho_2\sigma_2 + \delta\rho_4\sigma_4,$$
$$\alpha\rho_1^2 + \gamma\rho_3^2 = \beta\rho_2^2 + \delta\rho_4^2,$$
$$\alpha\sigma_1^2 + \gamma\sigma_3^2 = \beta\sigma_2^2 + \delta\sigma_4^2.$$

Square both sides of the first,* multiply together the two last, and subtract; $\alpha\gamma\,(\rho_1\sigma_3-\rho_3\sigma_1)^2=\beta\delta\,(\rho_2\sigma_4-\rho_4\sigma_2)^2$;

$$\therefore\ (\rho_1\sigma_3-\rho_3\sigma_1)\sqrt{\alpha\gamma}=\pm(\rho_2\sigma_4-\rho_4\sigma_2)\sqrt{\beta\delta}\ \ldots\ldots\ldots\ldots(7).$$

Eliminating σ_4 from equations (6) and (7), and reducing, we shall find that equation (6), when expressed in terms of the tripolar vectors σ_1, σ_2, σ_3, represents the two curves

$$\alpha\rho_1\,(\rho_1\sigma_2-\rho_2\sigma_1)+\gamma\rho_3(\rho_3\sigma_2-\rho_2\sigma_3)=\pm\sqrt{\frac{\alpha\gamma\delta}{\beta}}\,\rho_4\,(\rho_3\sigma_1-\rho_1\sigma_3)\ldots\ldots\ (8);$$

that is, two bicircular ovals through R, foci 1, 2, 3; and 4 will also be a focus, for σ_1 might have been eliminated instead of σ_4.

8. If we draw the diagonals 13, 24 of the quadrilateral 1234, and call their four segments u_1, u_2, u_3, u_4 (u_1 being the segment most remote from the focus 1), the equation (8) of the two bicircular ovals, with 1, 2, 3, 4 as foci, and meeting in a given point R, may be thus expressed (1, 2, 3 being the poles of reference),

$$u_1\rho_1\,(\rho_2\sigma_1-\rho_1\sigma_2)+u_3\rho_3\,(\rho_2\sigma_3-\rho_3\sigma_2)=\pm\,u_4\rho_4\,(\rho_3\sigma_1-\rho_1\sigma_3)\ \ldots\ldots\ (9).$$

If $\rho_1=\rho_2=\rho_3=\rho_4$, we have the two serpentine circular cubics whose foci are 1, 2, 3, 4 [equation (3)]; as $u_1u_3=u_2u_4$, their tripolar equations are

$$\frac{\sigma_1-\sigma_2}{u_3}+\frac{\sigma_3-\sigma_2}{u_1}\pm\frac{\sigma_1-\sigma_3}{u_2}=0\dagger\ \ldots\ldots\ldots\ldots\ldots(10).$$

The equations of the ovals of the same two circular cubics will be

$$\left.\begin{aligned}\frac{\sigma_1-\sigma_2}{u_3}-\frac{\sigma_3+\sigma_2}{u_1}+\frac{\sigma_1+\sigma_3}{u_2}&=0\\ \frac{\sigma_1+\sigma_2}{u_3}-\frac{\sigma_3-\sigma_2}{u_1}-\frac{\sigma_1+\sigma_3}{u_2}&=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(11).$$

These pass through the intersection of the diagonals (u_3, u_4, u_1, u_2); their equations only differ from (10) by σ_3 changing sign in the first, and σ_1 in the second.

9. If P be a point (r, s) referred to two fixed poles A, B, and P′ a consecutive point $(r+dr, s+ds)$, then

$$\frac{rds-sdr}{\text{AB}}=\text{projection of PP′ on the circle PAB.}$$

The proof of this property (by the infinitesimal method) I leave to the reader.

* By squaring we introduce the new factor $\alpha\rho_1\sigma_1+\beta\rho_2\sigma_2+\gamma\rho_3\sigma_3+\delta\rho_4\sigma_4=0$; but as all the terms are positive, this is an impossible locus, so that all *real* curves we arrive at by the process are contained in equation (6).

† If 1, 2, 3, 4 form a rectangle, (3) becomes $\sigma_1-\sigma_2+\sigma_3-\sigma_4=0$, and (10) the equivalent tripolar equation reduces to $\sigma_1=\sigma_2$, and $\sigma_3=\sigma_2$. Hence the circular cubics here become two straight lines through the centre, parallel to the sides of the rectangle.

Referring now to equation (7), we shall easily see that

$$\frac{\rho_1\sigma_3-\rho_3\sigma_1}{(13)} = \pm\frac{\rho_2\sigma_4-\rho_4\sigma_2}{(24)}.$$

Take* the point S $(\sigma_1\sigma_2\sigma_3\sigma_4)$ on one of these bicircular ovals, infinitely near R, their intersection,

$$\frac{\rho_1 d\rho_3-\rho_3 d\rho_1}{(13)} = \frac{\rho_2 d\rho_4-\rho_4 d\rho_2}{(24)}.$$

Hence the projections of RS on the circles R13, R24 are equal: so that the oval is equally inclined to these circles. For the other oval, the projections are also equal with opposite signs, so that it bisects the supplemental angle of the same two circles. Hence

The arc of any bicircular quartic (*foci* 1, 2, 3, 4), *at any point* R, *bisects the angle made by the circles* R13, R24.*

This beautiful theorem is due to Mr. Clifford. I believe he has not published any solution to it. The analogy to the conics and Cartesians is obvious.

Also:—*All confocal bicircular quartics cut each other orthogonally.*

It is also true that the curve is equally inclined to the circles R12, R34, and to R14, R23: this may be shown by elementary geometry to be a consequence of the preceding property; it may also be shown in the same manner as the latter has been (certain precautions being observed).

10. Draw the circles R12, R34; let R′ be the second intersection of these circles; produce 21, 34 to meet in V, then V, R, R′ are collinear.

It is easy to prove that $\frac{\rho_1\rho_1'}{\rho_2\rho_2'} = \frac{V1}{V2} = \frac{\beta}{\alpha}$;

hence $\alpha\rho_1\rho_1' = \beta\rho_2\rho_2'$; also $\gamma\rho_3\rho_3' = \delta\rho_4\rho_4'$.

As these values of ρ_1', ρ_2', ρ_3', ρ_4' satisfy eq. (6), it follows that R′ is on one of the ovals through R: also by some reduction it will be found that the one in question will be given by the negative sign in eq. (8); and this oval, it may be shown by the method explained further on, has either 1, 2 or else 3, 4 as its internal foci.

As VR . VR′ = V1 . V2, or V3 . V4, it is clear that the rectangle of the segments of any transversal from V, cutting the oval in two points, is constant.†

Hence, if the line of the two internal foci meet that of the two external in V, *a bicircular oval is anallagmatic with regard to the circle from* V *as centre, cutting the primary circle orthogonally.*

I will call this the secondary circle. Hence every bicircular oval is

* Mr. Hart has remarked to me that this property may be simply deduced from the corresponding property of the Cartesian ovals, by inverting the whole figure from one of the four foci.

† The transversal may meet the oval in four points; but each of them will have one of the others as its inverse.

doubly anallagmatic; with regard to the primary circle (Art. 4), and also to this circle.

The same properties as in Art. 4 of course hold for this secondary circle, which therefore cuts the system of confocal ovals orthogonally in two points of greatest or least curvature.

There is another secondary circle (centre W, the intersection of 14, 23), cutting the primary circle and the other secondary orthogonally. It is easy to show that these two secondary circles themselves are bicircular ovals belonging to the confocal system.* As in the case of Cartesians, the system of confocal ovals divides into two families, a secondary circle belonging to each. No two of the same family can meet, but every oval is cut (orthogonally) by all those of the opposite family.

Two *conjugate ovals* will always be on different sides of the secondary circle to whose family they belong, one inside and one outside, and are inverse to each other with regard to that circle.

11. The results we have arrived at establish a very striking analogy between the bicircular oval and the ellipse. As the ellipse has two axes at right angles, so the oval has what may be called two *circular axes*, also at right angles, viz. the arcs (AA′, BB′), which it intercepts on the primary and secondary circles; these axes cut the oval orthogonally, in four points, all of maximum or minimum curvature, as in the ellipse. In the ellipse, every point is accompanied by its reflection, or inverse, with regard to either axis; in the oval, every point has its inverse with regard to either circular axis. The oval, again, has two real foci on its primary axis AA′, being inverses with regard to the secondary axis BB′. The parallel holds also for the double contact circles, and doubtless extends to many other properties. I regret that want of leisure obliges me to leave many interesting details unexamined for the present.

12. No bicircular oval can pass through a focus, unless one of the three remaining foci coincide with it.

Suppose one of the pair of ovals through any point R, viz.

$$\alpha\rho_1\sigma_1-\beta\rho_2\sigma_2+\gamma\rho_3\sigma_3-\delta\rho_4\sigma_4=0,$$

to pass through the focus 1. Put $12=a$, $23=b$, $34=c$, $41=d$, $13=\Delta$, $2=\Delta'$: we shall have

$$-\beta\rho_2 a+\gamma\rho_3\Delta-\delta\rho_4 d=0.$$

Now, as the area of a triangle varies as the product of its sides, if the

* By finding their intersections, it is easy to show that the equation

$$\sigma_1\sqrt{\alpha}-\sigma_2\sqrt{\beta}+\sigma_3\sqrt{\gamma}-\sigma_4\sqrt{\delta}=0$$

represents the two secondary circles. That they belong to the system of ovals, is obvious, as for any point on the first $\sigma_1\sqrt{\alpha}=\sigma_2\sqrt{\beta}$, and also $\sigma_3\sqrt{\gamma}=\sigma_4\sqrt{\delta}$: so a linear relation holds for any three of σ_1, σ_2, σ_3, σ_4.

circumscribed circle be given, we may put $cd\Delta$ for β, $ad\Delta'$ for γ, $ab\Delta$ for δ; hence $$ad\Delta\ (c\rho_2+b\rho_4-\Delta'\rho_3)=0.$$

The quantity in the brackets being positive,* we must have $a=0$, $d=0$, or $\Delta=0$ (the latter involving one of the two former). Hence the focus 2 or 4 coincides with 1.

13. The complete bicircular quartic can consist of two ovals only. From rectangular coordinates we know that a circle can meet it in four points only. Hence, if it comprised three ovals, we might take a point inside each, and make a circle pass through these three points, which must cut the quartic in six points. Hence, if R, S, T be three foci, one of the three ovals

$$\left.\begin{aligned} -l\rho+m\sigma+n\tau&=0\\ l\rho-m\sigma+n\tau&=0\\ l\rho+m\sigma-n\tau&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

[the only possible combinations in eq. (4)] is imaginary. The only exceptional case is when the three ovals in (12) reduce to the three arcs RS, ST, TR of the primary circle.

The two conjugate ovals may be external to each other, or one inside the other: they never can meet (if the four foci are distinct); this is evident from the forms of the equations (12), for if two meet we must have $\rho=0$, $\sigma=0$, or $\tau=0$: which is impossible (Art. 12). Or, if we suppose two conjugates to meet, a circle will pass through their two intersections, and a third point on one of them, thus giving five intersections with the quartic.

14. *Two foci of any bicircular oval are internal, and the other two external.* Let $$U=l\rho+m\sigma-n\tau=0$$ be the oval. If the vectors of any point $(\rho\sigma\tau)$ render U positive, and those of another point render U negative, the points must be one outside and one inside.

Now, putting ST$=a$, TR$=b$, RS$=c$, we have

$$\begin{aligned} \text{for the focus R, } U_1&=mc-nb,\\ \text{for S, } U_2&=lc-na,\\ \text{for T, } U_3&=lb+ma. \end{aligned}$$

We see that U_3 is positive; now U_1 or U_2 is negative, for if not, we have $$\frac{m}{n}>\frac{b}{c},\ \text{and}\ \frac{l}{n}>\frac{a}{c};$$

therefore, for any point on the oval,

* It is known that, if A, B, D, C be any four points, we have AB . CD + AC . DB > AD . BC; unless they are concyclic, when we have Ptolemy's theorem. If the point $(\rho_1\rho_2\rho_3\rho_4)$ be on the primary circle, we might have $c\rho_2+b\rho_4-\Delta'\rho_3=0$; so that one of the bicircular ovals passes through 1: but this oval is the primary circle itself, or rather an arc of it.

$$\tau = \frac{l}{n}\rho + \frac{m}{n}\sigma > \frac{a}{c}\rho + \frac{b}{c}\sigma;$$

$$\therefore c\tau > a\rho + b\sigma,$$

which is impossible, for $a\rho + b\sigma > c\tau$ (Art. 12, note).

Hence, for any three foci, two are outside the oval and one inside, or two inside and one outside; whence the theorem follows.

If the bicircular quartic consists of two ovals external to each other, two foci are within each. If one conjugate oval is within the other, two foci are within the inner, and the remaining two outside the outer.

The centre of the primary circle is external to all ovals; as, if ρ, σ, τ are equal, U gives the same sign as for infinity (when ρ, σ, τ are also equal). This is also clear from the anallagmatic property. The only exception is for serpentine circular cubics—the only case when an oval opens out into an infinite curve.

15. To determine the condition that the bicircular quartic

$$\pm l\rho \pm m\sigma \pm n\tau = 0$$

shall consist of two ovals, one within the other.

Let the real ovals be

$$U = l\rho + m\sigma - n\tau = 0,$$
$$V = l\rho - m\sigma + n\tau = 0;$$

then any focus, R, is inside both, or else outside both. Now, if it is outside both, the function U, for the point R, has the same sign as for an infinitely remote point; and likewise for V; hence,

$mc - nb$ has the same sign as $l + m - n$,

and $-mc + nb$ has the same sign as $l - m + n$.

If R be *inside* both, the signs will be *different*, instead of the same.

Whether R is outside or inside both, then

$$-(mc - nb)^2 \text{ has the same sign as } l^2 - (m-n)^2.$$

Hence $$l^2 - (m-n)^2 < 0, \quad \therefore l < m \sim n.$$

We have, then, $$m > l + n, \text{ or } n > l + m.$$

If $l > m + n$, the oval $-l\rho + m\sigma + n\tau = 0$ is always real,* and therefore one of those I have called U, V is imaginary; hence it is easy to see that the same proof applies in this case, and the ovals will be one within the other. If, therefore, in $\pm l\rho \pm m\sigma \pm n\tau = 0$ *any one of the multipliers* l, m, n *exceeds the sum of the other two, one of the ovals is*

* Because the focus R gives the function positive, viz. $mc + nb$; now S gives $na - lc$, and T gives $ma - lb$: one of these must be negative, as otherwise we should have $(m+n)\,a > l\,(b+c)$, and therefore $a > b + c$. Since, then, a real point gives the function negative, and another gives it positive, the function must vanish at some point on any curve joining the two points. The conditions that $l\rho = m\sigma + n\tau$ be imaginary, are also easily seen to be

$$\frac{m}{l} > \frac{b}{a}, \text{ and } \frac{n}{l} > \frac{c}{a}.$$

within the other. If each multiplier is less than the sum of the other two, the ovals are external to each other.

If one constant, as l, is *equal* to the sum of the other two, we have the limiting case of the circular cubic, one oval becoming infinite.

16. To determine the conditions that the bicircular quartic shall consist of but one oval.

In this case, two ovals, as $\left.\begin{aligned} l\rho &= m\sigma + n\tau \\ m\sigma &= l\rho + n\tau \end{aligned}\right\}$,

are imaginary. Hence (see Note, Art. 15), for the first, we have

$$\frac{m}{l} > \frac{b}{a}, \text{ and } \frac{n}{l} > \frac{c}{a};$$

for the second,

$$\frac{l}{m} > \frac{a}{b}, \text{ and } \frac{n}{m} > \frac{c}{b}.$$

Now we cannot have $\frac{m}{l} > \frac{b}{a}$, and $\frac{l}{m} > \frac{a}{b}$. Hence, strictly speaking two ovals cannot be imaginary. Nevertheless, when $\frac{l}{m} = \frac{a}{b}$, if the two remaining conditions above are satisfied, so that the ovals become

$$\left.\begin{aligned} a\rho &= b\sigma + N\tau \\ b\sigma &= a\rho + N\tau \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

where $N > c$, they each consist of one real point only, viz. the focus T.

The remaining oval $a\rho + b\sigma = N\tau$ .....(14)

is real. In this case, then, the quartic consists of one real oval and the focus T as a *conjugate point*. Of course the fourth focus must coincide with T (Art. 12).

If $N > c$, (14) becomes imaginary, and the two ovals in (13) are real. They cross each other at the double focus T, the pair forming a single curve with a double point. The conditions laid down in Art. 15 will determine whether they lie one inside the other (like the *limaçon* in Cartesians), or resemble a figure of 8 distorted.

We have stated (Art. 10) that two conjugate ovals are inverse with regard to one of the secondary circles. This, of course, shows that, when the four foci are distinct, the quartic can never consist of one oval only; unless in the case where the two conjugates coincide, or merge into the secondary circle, which thus may be viewed as a *double oval*, being its own conjugate. The other extreme limit to which a pair of conjugates may tend is when they degenerate into the arcs 12, 34 of the primary circle, as an ellipse degenerates into the line joining its foci. A pair of conjugates of the other family degenerate into the arcs 14, 23.

17. It may be shown, from the anallagmatic property, that if a bicircular oval have an indentation, the double tangent passes through

the centre of the primary circle, or else through the centre of the secondary circle which meets the oval.

18. It is easy to see (as in Art. 4) that all *circles* represented by

$$A\rho^2+B\sigma^2+C\tau^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

are orthogonal to the primary circle RST.

If $A+B+C=0$, equation (15) represents *a diameter of the primary circle.*

Let $(\rho'\sigma'\tau')$ be any point on the circle (15):—It is easy to show that the bicircular oval* $A\rho\rho'+B\sigma\sigma'+C\tau\tau'=0$ .....(16)

touches the circle at that point: and also at the inverse point.

Or, if $l\rho+m\sigma+n\tau=0$ be any bicircular oval, and $(\rho'\sigma'\tau')$ any point on it, *the equation of the circle touching the curve there, and at the inverse point, is*

$$l\frac{\rho^2}{\rho'}+m\frac{\sigma^2}{\sigma'}+n\frac{\tau^2}{\tau'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

This double-contact circle becomes a straight line through the centre of RST, *i.e. a double tangent to the oval,* if $(\rho'\sigma'\tau')$ satisfy

$$\frac{l}{\rho'}+\frac{m}{\sigma'}+\frac{n}{\tau'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Hence, if $(\rho'\sigma'\tau')$ be a point of contact of such a double tangent, then a point $(\rho''\sigma''\tau'')$ whose vectors are *inversely proportional* to ρ', σ', τ', will also lie on the oval (it may not be real), and $(\rho''\sigma''\tau'')$ will also be a contact of another double tangent. Hence a bicircular oval may have two double tangents from the centre.

19. Hence, if any Cassinian oval pass through three points R, S, T, and F, F′ be its foci, a bicircular oval through FF′, with R, S, T as foci, has double tangents at F and F′.

Also if a Cassinian oval pass through four concyclic points 1, 2, 3, 4, a bicircular oval (foci 1, 2, 3, 4) can be drawn through the two foci of the Cassinian (as is plain from eq. 6), and the tangents to it at those two points are double tangents.

Hence the locus of foci of all Cassinians through four concyclic points is identical with the locus of the contacts of double tangents to the system of bicircular quartics whose foci are the given four points. This locus will be found to consist of the two curves

$$c\rho_3\rho_4(\rho_1^2-\rho_2^2)=\pm a\rho_1\rho_2(\rho_3^2-\rho_4^2),$$

where $a=(12)$, $c=(34)$. Together, they constitute a sextic curve in rectangular coordinates. An easy method of drawing the curves is deducible from the above equation.

* Several theorems analogous to those on Cartesians in Art. 18 of my Paper (*Proceedings*, No. 6) will also hold.

20. The following method enables us to trace a bicircular quartic, given the four foci 1, 2, 3, 4, and one point R on the curve.

Draw the circles 1R3, 2R4; draw *any* circle touching the arcs R1, R2 [but not meeting the circle (1234)]; find the pair of points which are *co-inverse** to this circle and (1234); both of these are points on the oval which bisects the angle 1R2.

To prove this, let T be the tangent from R to this circle; t_1, t_2, t_3, t_4 the tangents to it from 1, 2, 3, 4; then, by Mr. Casey's extension of Ptolemy's theorem, it will be easy to see that

$$(13)\ T = \rho_3 t_1 - \rho_1 t_3, \text{ and } (24)\ T = \rho_2 t_4 - \rho_4 t_2;$$

$$\therefore \frac{\rho_3 t_1 - \rho_1 t_3}{(13)} = \frac{\rho_2 t_4 - \rho_4 t_2}{(24)}.$$

But if S be one of the co-inverse points above, σ_1, σ_2, σ_3, σ_4 are proportional to t_1, t_2, t_3, t_4: hence (eq. 7) the theorem follows.

If a vertex, A, be given, the curve may be thus drawn:—Suppose A to lie on the arc 12; draw *any* circle touching (1234) internally at A; draw two circles to touch it, one through 14, the other through 23; they intersect each other in two points on the oval.

The four vertices of the two bicircular ovals through any point R will be the points in which (1234) is touched by four circles inscribed in the four curvilinear triangles into which (1234) is divided by the circles 1R3, 2R4.

21. It is worth remarking, that (putting ST = a, TR = b, RS = c) the bicircular quartics

$$a \cos \lambda . \rho \pm b \cos \mu . \sigma \pm c \cos \nu . \tau = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

and

$$a \sin \lambda . \rho \pm b \sin \mu . \sigma \pm c \sin \nu . \tau = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

(where λ, μ, ν are any angles) cut each other orthogonally, and have their fourth focus common.

I have been obliged, in this Paper, to leave much to be supplied by the reader. Had I given all the proofs in full, and entered into exhaustive discussions of several points which I have merely glanced at, the limits at my disposal would have been far exceeded. I hope, however, that, with some attention, the reader will find it possible to follow the reasonings I have used. It will be observed that I confine myself to bicircular quartics with four real concyclic foci. The other class of bicircular quartics, which have two real foci and two imaginary on one circle, and two real with two imaginary on another circle, differs in form, and in many properties, from that we have been examining, and requires separate and careful study. Of course it includes the species of Cartesian which has two imaginary foci; few of whose properties are as yet known.

* Any two circles which do not meet have a pair of points which are inverse to each other with regard to *either* circle. These I call *co-inverse* points. They are easily found, in various ways.

A communication was made "On Angles and Parallels": by Mr. A. J. Ellis, F.R.S.

Prof. Hirst gave a description of certain Degenerate Forms of Quartic Curves, satisfying given conditions, to which Chasles had recently drawn attention. He remarked that an interesting illustration of Chasles' views was furnished by Mr. Crofton's Paper. Mr. Crofton, in fact, had shown that one of the systems of quadri-confocal bicircular quartics consisted of *two circular arcs, considered as doubled, and bounded each by two of the four foci.* This was entirely in accordance with Chasles' theory as given in his recent Paper in the "Comptes Rendus."

June 27, 1867.

A. DE MORGAN, Esq., Vice-President, in the Chair.

Messrs. Martin Gardiner and Archibald Smith, M.A., F.R.S., were elected Members.

Mr. Rumsey read a Paper "On Ancient Modes of Calculation prescribed by the Mohammedan Law of Inheritance."

Prof. Hirst communicated a Paper "On the determination of Double Entities in Uniquadric Homographics," by Mr. Gardiner.

Prof. Clerk-Maxwell exhibited some Stereoscopic Engravings representing the Lines of Curvature on various surfaces.

November 14*th*, 1867.

Special General Meeting, held at Burlington House.

Prof. SYLVESTER, President, in the Chair.

At this Meeting it was resolved, that the Annual Subscription be raised from ten shillings to one guinea, beginning with the subscription due on that day for the ensuing Session 1867–8, and that the requisite alteration be made in *Rule* 19. It was also resolved unanimously, that the following alteration be made in *Rule* 30, by substituting for the words, "both inclusive. Should the fourth Thursday in December, however, fall on the 25th of that month," the words, "except in December, when the Meeting shall be held on the second Thursday."

The business of this Meeting being then concluded, there was held

The Annual General Meeting.

The Reports of the Treasurer and Secretaries were read and adopted.

From the Report of the Secretaries, it appeared that the Society had,

since the last General Meeting, held November 8th, 1866, increased in number from 94 to 104 Members. One honorary foreign member, M. Chasles, had also been elected under the rules passed at the beginning of the Session.

The recent death of the late Secretary, Mr. George Campbell De Morgan, and the great loss which the Society had thereby sustained, was referred to. The fact was also mentioned, that he, in conjunction with Mr. Ranyard, had formed the idea of establishing this Society, and had taken the preliminary steps towards its formation.

The communications which had been made to the Society during the past Session were as follows:—

"On Harmonics in Space:" by Mr. W. K. Clifford.

"On Analogies in the Parallelopiped to properties of the Parallelogram:" by Mr. J. J. Walker.

"On the Development of a certain class of Functions:" by Mr. G. C. De Morgan.

"On the Order of certain systems of Equations:" by Mr. Samuel Roberts.

"On the inscription of a Polygon in a ruled Quadric:" by Rev. R. Townsend, communicated by Prof. Hirst.

"On extensions of Möbius and Feuerbach's theorems relating to a variable and four fixed points, with applications to the proof of properties of Focal Curves:" by Mr. Crofton.

"On properties of Homographic Systems of Points on a ruled Quadric:" by Rev. R. Townsend, communicated by Prof. Hirst.

"On the Law of Unicursal Derivation of Point from Point on Cubic Curves:" by Prof. Sylvester.

"On some Theorems connected with Prof. Sylvester's Theory of Derivation:" by Prof. Cayley.

"On Residuals and Opposites in Cubic Curves:" by Prof. Sylvester.

"A new Proof of the Rectangle of Forces:" by Mr. J. J. Walker.

"On the Conic Octagram:" by Prof. De Morgan.

"On a method of finding the ninth of the Points of Intersection of two Cubics, seven of these points being fixed, whilst the eighth moves on a straight line:" by Prof. Cayley.

"On an Arithmetical Theorem applicable to the proof of an extension of Legendre's Law of Reciprocity:" by Mr. Jenkins.

"On certain properties of Bicircular Quartics:" by Mr. Crofton.

"On Angles and Parallels:" by Mr. Ellis.

"On Ancient Modes of Calculation prescribed by the Mohammedan Law of Inheritance:" by Mr. Almaric Rumsey.

"On the determination of Double Entities in Uniquadric Surfaces:" by Mr. Gardiner, communicated by Prof. Hirst.

Besides these, many lesser communications had been made.

The same Mathematical Journals had been subscribed for as in the preceding Session.

The Proceedings of the Society had been sent to the following Societies and Institutions besides those enumerated in the last Report: viz., the Akademie der Wissenschaften of Berlin, and the Istituto Lombardo of Milan; and also to Prof. Cremona, an Editor of the Annali di Matematica, of the R. Istituto Tecnico di Milano.

In return for these, the Society had received the Proceedings of the Royal Society, the Journal of the Institute of Actuaries, the Memorie, Rendiconti, and Solenni Adunanze of the Istituto Lombardo, the Annali di Matematica from Prof. Cremona, the Monatsberichte from the Akademie der Wissenschaften, the Journal de l'École Polytechnique from the Director of the École Polytechnique, and the Bulletin of the Société Philomathique.

The Chemical Society had kindly renewed its permission for the Meetings of this Society to be held in their rooms at Burlington House.

The following presents had been received during the past Session:—

Some models of Surfaces of the Fourth Order illustrative of Prof. Plücker's new Theory of Complexes, from Prof. Hirst.

A Photographic Album, to hold 192 Cartes-de-Visite, from Mr. Tucker.

Eighteen Mathematical Text-books, from Mr. Spottiswoode.

The following Officers and Council were elected to serve during the ensuing year:—President, Prof. Sylvester; Vice-Presidents, Prof. H. J. S. Smith, and Messrs. A. De Morgan and Spottiswoode; Treasurer, Prof. Hirst; Secretaries, Messrs. Jenkins and Tucker; other Members of Council, Prof. Cayley and Price, Messrs. T. Cotterill, Crofton, Ellis, Ingleby, Nesbitt, Roberts, and Stirling.

Arnold William Reinold, Esq., M.A., Fellow of Merton College, Oxford, was elected a Member.

Votes of thanks were carried to the Chemical Society for the renewed use of their rooms; to Dr. Sharpey for the continued use of shelves in his room at University College; and to the retiring Vice-President and Members of Council for their services.

After the conclusion of business, Mr. S. Roberts gave "The General Construction of 16 chords, which connect in threes the 12 points of contact of tangents drawn to a curve of the third degree from three collinear points upon it," and illustrated by a model and stereographs, the figure in space of three dimensions (viz., the projection of a complete parallelopiped), which is represented in perspective by the chords *in plano*. Mr. Crofton announced the results obtained in some theorems of a new character in *Local Probability*, from some of which it would appear that this Theory is likely to be of use in that of Definite Integrals.

November 28th, 1867.

Prof. HIRST, Treasurer, and subsequently W. SPOTTISWOODE, Esq., Vice-President, in the Chair.

The following gentlemen were elected Members:—

J. McDowell, Esq., M.A., F.R.A.S., Pembroke College, Cambridge; C. E. F. Nash, Esq., M.A., Pembroke College, Oxford; and Sir William Thomson, LL.D., F.R.S., Prof. of Natural Philosophy in Glasgow University.

A vote of thanks was carried to the Reale Istituto Lombardo for the present of their publications.

The following communications were made:—

Prof. Hirst, "On the representation of Points in Space by Triplets of Points on a Line." Following a mode of demonstration analogous to one given by Hesse,* it may be shown that, when thus represented, the points in space which correspond to the several points of the line, regarded as *triple points*, lie, in general, on a cubic curve of double curvature. The triplets of collinear points which represent the several points of any right line in space constitute an involution of the third order, as defined by De Jonquières.† Such an involution contains six triplets, each of which consists of a single and a double point; the points in space which correspond to these six triplets are the intersections of the right line, represented by the involution, and the *developable osculatrix* of the above-named cubic in space.

This mode of representation, by means of which the Geometry of Space is reduced to the Geometry of a Right Line, may, of course, be realized by taking any cubic in space (C) and a fixed tangent thereto T. The three osculating planes to the cubic from any point α in space will intersect the tangent T in the three points α_1, α_2, α_3 which may be regarded as the triplet representing α; for, conversely, any three points α_1, α_2, α_3 being taken on T, only one osculating plane, not containing T, can be drawn through each point, and therefore the three osculating planes will determine, by their intersection, a unique point α in space. If the point α lie on the cubic, the three osculating planes, and hence the three representative points α_1, α_2, α_3, will coincide; if α lie on a tangent to the cubic, two of the three osculating planes, and hence two of the three representative points on T, will coincide, &c.

The author proposes to develope further consequences of this mode of representation in a future paper.

* In his paper, entitled "Ein Uebertragungsprincip," in Crelle's Journal, Bd. 66, p. 15.

† See his memoir on a *Généralisation de la Théorie de l'Involution*, published in the *Annali di Matematica*, t. 2, p. 86. (Rome, 1859.)

Mr. T. Cotterill mentioned a property of 6 points on a plane, or sphere, of which Pascal's theorem is a particular case. "If 6 points are taken in a plane and the three quadrilaterals formed from them which by the intersection of their diagonals give one set of Pascal points, then the product of the areas of the quadrilaterals and the Pascal triangle is invariable in magnitude, whatever be the order in which the points are taken: and is moreover equal to that function of the areas of the triangles formed from the six points, which vanishes when any two of the points are supposed to coincide. The same is true (mutatis mutandis) for the sphere."

Mr. J. J. Walker made a short communication "On the discrimination of the roots of the Biquadratic by inspection of the signs of the coefficients and discriminant of Euler's Reducing Cubic," in the course of which he pointed out, that the roots of the cubic may be (1) all real and positive, (2) all real and two negative, (3) two imaginary. In the first case, the roots of the Biquadratic are all real; in the second, all imaginary; in the third, two real and two imaginary. The signs of the coefficients and of the discriminant of the cubic afford ready tests of the above cases.

Mr. Walker then submitted the following "Note on Partial Invariants." "Besides the recognized invariants of a binary form, there are functions of the coefficients which are invariants when one only of the variables is supposed to be transformed, which are of equal importance in the theory of Algebraic Equations." It was proposed to recognize these too formally, and to term them "Partial" as distinguished from "Total" Invariants. If the coefficients, reckoned from either end, enter symmetrically into a Partial, it is also a Total Invariant.

A short account of a system of coordinates of lines in space, peculiarly adapted to the subject of Statics, was then given by Mr. Spottiswoode, V.P.

The following communications were unavoidably omitted in the account of the proceedings of the meeting held on November 14:—

Prof. Brayley proposed to draw the attention of mathematicians, with reference to classification of organised beings, towards a question of arrangement, which is in truth a logical problem of the widest range. Given a number of individuals, finite or infinite, possessing each a number of characteristics, finite or infinite, in all intensities of degree; required a distribution under which proximity of character may be detected by proximity of position. This problem, though in all its generality it may long be unsolved, has been, as Mr. Brayley points out, in the aspiration of some naturalists; and it may be useful to point the attention of mathematicians towards it, in the hope that at least some of its simpler cases may be forthwith considered. Mr. Brayley concluded by expressing his opinion that many important subjects relating

to the natural distribution *inter se* of both animals and plants, now required for their further progress, and were adapted to receive, the aid of mathematical analysis.—[Prof. Brayley's problem is founded on an investigation of W. Sharp Macleay's views of natural arrangement, by the late H. T. Colebrooke.* These views of Mr. Macleay's will be found in his paper, "The Circular and Quinary Distribution of Animated Nature," published in his "Horæ Entomologicæ;" also consult the 4th volume of the "Zoological Journal," published in 1828.]

Prof. Sylvester then made a brief allusion to a problem of arrangement connected with a new method of tesselation, which had led him to discover the following theorem in the partition of numbers:—*The square root of the sum, or else of the double of the sum, of any complete progression of binomial coefficients may always be obtained from the sum itself, on changing the signs of certain of the terms from positive to negative.*

The theorem, enunciated under a slightly different form, more convenient for the purposes of verification, was illustrated by numerical examples.

December 12th, 1867.

Prof. SYLVESTER, President, in the Chair.

Sir. J. G. Shaw-Lefevre, K.C.B., and Mr. W. S. B. Woolhouse, F.R.A.S., were elected members.

The following papers were read:—

On the Logarithms of Imaginary Quantities.

By Prof. Cayley, F.R.S., V.P.

The theory of the logarithms of imaginary quantities admits of a remarkably simple representation.

Let P denote at pleasure the imaginary quantity $x+iy$, or else the point the coordinates of which are (x, y); viz., P regarded as a quantity will denote $x+iy$; but we may also speak of the point P.

Writing thus $\quad P = x+iy,$

and similarly $\quad P' = x'+iy',$

we have of course $\quad \frac{P}{P'} = \frac{x+iy}{x'+iy'},$

an imaginary quantity $X+iY$; and the point $\frac{P}{P'}$ will be the point the coordinates of which are (X, Y).

We have $\quad P = re^{i\theta},$

* Translator, from the Sanskrit, of the Hindu Algebra; President of the (now Royal) Astronomical Society.

viz., r is $= \sqrt{x^2+y^2}$, the radical being positive, and θ is an arc such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}},$$

and moreover θ may be taken to be an arc between the limits $-\pi$, $+\pi$. The arc so defined may be denoted by $\tan^{-1}\frac{y}{x}$, so that we have $\theta = \tan^{-1}\frac{y}{x}$.

It is to be observed that θ has always a determinate unique value, except in the single case $y=0$, x negative, where we have indeterminately $\theta = \pm\pi$.

It is further to be remarked that, taking A for the origin of coordinates, we have θ = angle xAP, considered as positive or as negative according as P lies above or below the axis of x.

Starting from the equation $\quad \mathrm{P} = re^{i\theta}$,

we have similarly $\quad \mathrm{P}' = r'e^{i\theta'}$,

and $\quad \frac{\mathrm{P}}{\mathrm{P}'} = \frac{r}{r'}e^{i\phi}$,

where ϕ is derived from $\frac{\mathrm{P}}{\mathrm{P}'}$ in the same way as θ from P, or θ' from P'.

Consequently $\quad e^{i(\theta-\theta'-\phi)} = 0$,

and therefore $\theta-\theta'-\phi$ a multiple of 2π, say

$$\theta-\theta'-\phi = 2m\pi,$$

and in this equation the value of m is determined by the limiting conditions above imposed on the values of θ, θ', ϕ. To see how this is, suppose in the first instance that the finite line or chord P'P, considered as drawn from P' to P, cuts the negative part of the axis of x upwards; P is then above, P' below, the axis of x; that is, θ, $-\theta'$ are each positive; and drawing the figure, it at once appears that the sum $\theta+(-\theta')$, that is $\theta-\theta'$, is a positive quantity greater than π. And in this case the angle ϕ will be equal to $2\pi-(\theta-\theta')$ taken negatively, that is, $\phi = -\{2\pi-(\theta-\theta')\}$, or $\theta-\theta'-\phi = 2\pi$. But, in like manner, if P'P cut the negative part of the axis of x downwards, P will be below, P' above, the axis of x; $-\theta$ and θ' are here each positive, and the figure shows that the sum $-\theta+\theta'$ is greater than π; and in this case the angle ϕ is $= 2\pi-(-\theta+\theta')$; that is, we have $\theta-\theta'-\phi = -2\pi$. In every other case, (that is, if the chord P'P either does not meet the axis of x, or if it meets the positive part of the axis of x,) $\theta-\theta'$ and ϕ are each in absolute magnitude less than π, and we have $\theta-\theta'-\phi = 0$. So that we see that, according as the chord P'P, considered as drawn from P' to P, meets the negative part of the axis of x upwards or

downwards, or as it does not meet the negative part of the axis of x, the value of $\theta-\theta'-\phi$ is $=2\pi$, $=-2\pi$, or $=0$.

Taking now $\log r$ to represent the real logarithm of the positive real quantity r, we may, as a definition of the logarithm of the imaginary quantity $\mathrm{P}\,(=x+iy)$, write

$$\log \mathrm{P} = \log r + i\theta.$$

The value so defined is of course one out of the infinite number of values of the logarithm, and it may for distinction be termed the "selected" value. In all that follows, the symbol "log" is to be understood to denote the selected value. We have

$$\log \mathrm{P} = \log r + i\theta,$$

and similarly

$$\log \mathrm{P}' = \log r' + i\theta',$$

and

$$\log \frac{\mathrm{P}}{\mathrm{P}'} = \log \frac{r}{r'} + i\phi.$$

Hence

$$\log \mathrm{P} - \log \mathrm{P}' = \log \frac{\mathrm{P}}{\mathrm{P}'} + i\,(\theta-\theta'-\phi),$$

so that, by what precedes, $\log \mathrm{P}-\log \mathrm{P}'$, if the chord $\mathrm{P}'\mathrm{P}$, considered as drawn from P' to P, cuts the negative part of the axis of x upwards, is $= \log \frac{\mathrm{P}}{\mathrm{P}'} + 2i\pi$; if the chord cuts the negative part of the axis of x downwards, it is $= \log \frac{\mathrm{P}}{\mathrm{P}'} - 2i\pi$, and in every other case it is $= \log \frac{\mathrm{P}}{\mathrm{P}'}$.

It is to be remarked that $\log \mathrm{P}$, as above defined, is a continuous function of $\mathrm{P}\,(=x+iy)$, with the single exception that, if the point P move from above to below or from below to above the negative part of the axis of x, the imaginary part of the logarithm changes from $+i\pi$ to $-i\pi$, or from $-i\pi$ to $+i\pi$, in the two cases respectively. And we are thus led to another mode of looking at the question.

Consider the integral

$$\int_{\mathrm{P}'}^{\mathrm{P}} \frac{dz}{z}.$$

The value of the integral may depend on the series of values assumed by the variable z as it passes from the limit $z=\mathrm{P}'$ to the limit $z=\mathrm{P}$, or say it may depend on the path of the variable z; in order to give the notation a precise signification, we must therefore fix the path of the variable z; and I do this by taking the path to be the right line $\mathrm{P}'\mathrm{P}$. Write now $z=\mathrm{P}'.u$, we have $\frac{dz}{z}=\frac{du}{u}$; $z=\mathrm{P}'$ gives $u=1$; $z=\mathrm{P}$ gives $u=\frac{\mathrm{P}}{\mathrm{P}'}$; and it is easy to see that, the path of z being along the right line P' to P, that of u is along the right line 1 to $\frac{\mathrm{P}}{\mathrm{P}'}$ (that is,

from the point the coordinates whereof are $x=1, y=0$, to the point $\frac{P}{P'}$).

We have thus $$\int_{P'}^{P} \frac{dz}{z} = \int_{1}^{P \div P'} \frac{du}{u},$$

the path in each case being a right line as above. The indefinite integral $\int \frac{du}{u} = \log u$; and as u passes from 1 to $\frac{P}{P'}$, there is no discontinuity in the value of $\log u$; the value of the right-hand side is thus $= \log \frac{P}{P'}$. As regards the left-hand side, the indefinite integral is in like manner $= \log z$; but here, if the chord P'P cuts the negative part of the axis of x, there is a discontinuity in the value of $\log z$, viz., if the chord P'P, considered as drawn from P' to P, cuts the negative part of the axis of x upwards, there is an abrupt change in the value of $\log z$ from $-i\pi$ to $+i\pi$; and similarly, if the chord cut the negative part of the axis of x downwards, there is an abrupt change from $+i\pi$ to $-i\pi$; in the former case, by taking the definite integral to be $\log P - \log P'$, we take its value too large by $2i\pi$, in the latter case we take it too small by $2i\pi$; that is, the true value of the definite integral is in the former case $= \log P - \log P' - 2i\pi$, in the latter case it is $= \log P - \log P' + 2i\pi$. But if the chord PP' does not cut the negative part of the axis of x, then there is not any discontinuity, and the true value of the definite integral is $= \log P - \log P'$. We have thus in the three cases respectively

$$\log P - \log P' = \log \frac{P}{P'} + 2i\pi,$$
$$= \log \frac{P}{P'} - 2i\pi,$$
$$= \log \frac{P}{P'},$$

which agrees with the previous results.

It may be remarked, that it is merely in consequence of the particular definition adopted that there is in the value of $\log P$ a discontinuity at the passage over the negative part of the axis of x; with a different definition of the logarithm, there would be a discontinuity at the passage over some other line from the origin; but a discontinuity somewhere there must be. For if, as above, the chord P'P meet the negative part of the axis of x, then forming a closed quadrilateral by joining by right lines the points 1 to P, P to P', P' to $\frac{P}{P'}$, and $\frac{P}{P'}$ to 1; the only side meeting the negative part of the axis of x is the side P'P; the integral $\int \frac{dz}{z}$, taken through the closed circuit in question, or say the integral

$$\left(\int_{\mathrm{P}}^{1}+\int_{\mathrm{P}'}^{\mathrm{P}}+\int_{\mathrm{P}\div\mathrm{P}'}^{\mathrm{P}'}+\int_{1}^{\mathrm{P}\div\mathrm{P}'}\right)\frac{dz}{z}$$

has, by what precedes, a value in consequence of the discontinuity in passing from P′ to P; viz., this is $=-2i\pi$ or $=2i\pi$, according as the chord P′P, considered as drawn from P′ to P, cuts the negative part of the axis of x upwards or downwards; but this value $-2i\pi$ or $+2i\pi$ must be altogether independent of the definition of the logarithm; whereas if, by any alteration in the definition, the discontinuity could be avoided, the value of the integral, instead of being as above, would be $=0$. The foregoing value $-2i\pi$ or $+2i\pi$ is in fact that of the integral taken along (in the one or the other direction) any closed curve surrounding the point $z=0$ for which the function $\frac{1}{z}$ under the integral sign becomes infinite: but in obtaining the value as above, no use is made of the principles relating to the integration of functions which thus become infinite.

The equation $\log \mathrm{P} = \log r + i\theta$

gives $\mathrm{P}^m = e^{m \log \mathrm{P}} = r^m e^{im\theta}$,

or say $(x+iy)^m = r^m e^{im\theta}$,

where, m being any real quantity whatever, r^m denotes the positive real value of r^m. We have thus a definition of the value of $(x+iy)^m$, and the value so defined may be called the selected value. And similarly, for an imaginary exponent $m=p+qi$, we have

$$\begin{aligned}(x+iy)^{p+qi} &= e^{(p+qi)(\log r+i\theta)}\\ &= e^{p\log r - q\theta + i(p\theta+q\log r)}\\ &= r^p e^{-q\theta}.e^{i(p\theta+q\log r)},\end{aligned}$$

which is the selected value of $(x+iy)^{p+qi}$.

It may be remarked, in illustration of the advantage (or rather the necessity) of having a selected value, that in an integral $\int \mathrm{Z}dz$, taken between given limits along a given path, it is necessary that we know, for the real or imaginary value of z corresponding to each point of the path, the value of the function Z; and consequently, if Z is a function involving $\log z$ or z^m, the indeterminateness which presents itself in these symbols (considered as belonging to a single value of z) is, so to speak, indefinitely multiplied, and $\int \mathrm{Z}dz$ is really an unmeaning combination of symbols, unless by selecting, as above or otherwise, a unique value of $\log z$ or z^m, we render the function to be integrated a determinate function of the variable.

After the reading of this paper, Prof. Cayley made some remarks "On the algorithmic relations of the Double Tangents of a Quartic Curve."

A further communication "On the Theory of Local Probability" was made by Mr. M. W. Crofton.

In this paper, which has since, in an enlarged form, been presented to the Royal Society, the author considers the question of applying the theory of probability to straight lines drawn at random in a plane. Hitherto, the subject of local probability has been almost entirely restricted to *points* taken at random. It is here shown that the true mathematical conception of a system of random straight lines may be thus obtained:—Conceive a plane ruled with an infinity of parallel equidistant lines, whose common infinitesimal distance is given; then let the system revolve through an infinitesimal angle $\delta\theta$; then through an equal angle, and so on, till the parallels return to their former position. By applying mathematical calculation to such a system of lines, various theorems are established, which form the basis of this new application of probability. The author also uses the method in deducing the values of several double integrals. The paper concludes by merely touching on the extension of the method to straight lines and planes, taken at random in space; remarking that such an inquiry is one of great interest and great difficulty, involving several new and intricate questions in the general theory of surfaces.

The following are some of the results:—

The measure of the number of random lines which meet a given closed convex contour is L, the length of the contour.

If the contour be non-convex, or not closed, the measure is the length of a string passed round the contour and tightly enveloping it.

Hence the probability that a random line meeting the contour shall also meet another closed convex contour, length l, wholly enclosed by the former, is $$P = \frac{l}{L}.$$

If the contour l lie wholly outside L, the probability is

$$P = \frac{X-Y}{L},$$

where X is the length of an endless string tightly enveloping the two contours and crossing between them, and Y that of one which envelopes them without crossing.

If the contour l intersect L in two or more points,

$$P = \frac{L+l-Y}{L},$$

where Y is an endless string tightly drawn round both.

If two lines are drawn at random across a contour L, the probability that their intersection shall lie within the contour is

$$P = \frac{2\pi\Omega}{L^2},$$

Ω being the space enclosed by L.

Few problems on random straight lines admit of solutions so general

as these. The principles, however, laid down in this paper suffice, generally speaking, to reduce any such question to one of mathematical calculation; though it is seldom soluble unless for particular forms of the contours. As examples of such problems, the author considers, among others, the following problems relating to a circular contour:—

A random point and a random straight line fall within a given circle; to find the chance of the line passing within a given distance of the point.

Three random lines meet a given circle; to find the chance that the triangle they form shall lie wholly within the circle.

Among the incidental applications of the theory to the integral calculus which present themselves in the course of the inquiry, the following theorems may be cited:—

If L be the length of any closed convex contour, enclosing a space Ω, and θ be the angle it subtends at any external point (x, y),

$$\iint (\theta - \sin\theta)\, dx\, dy = \tfrac{1}{2}\mathrm{L}^2 - \pi\Omega :$$

the integral extending over all the external surface of the plane.

If we trace any external contour, so related to L that any tangent to L cuts off a constant area from it, and if Δ be the difference of the parts into which the annular space between the two contours is divided by any tangent to L, then

$$\iint \theta\, dx\, dy = \pi\Delta ;$$

the integral extending over all the annular space.

If we pass an endless string, of length Y, round the contour, and trace out an external contour by means of a pencil which always keeps the string stretched, then for the annular enclosed space

$$\iint \sin\theta\, dx\, dy = \mathrm{L}\,(\mathrm{Y} - \mathrm{L}).$$

An example of this theorem is furnished by two confocal ellipses. In the case of the preceding, we may take two concentric similar ellipses; if we suppose the outer to cover double the area of the inner, we may deduce the analytical integral

$$\iint \tan^{-1}\left(\frac{2\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{x^2 + y^2 - a^2 - b^2}\right) dx\, dy = 2\pi ab ;$$

the limits being given by $\quad 2 > \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} > 1.$

With regard to straight lines drawn at random in space, it is shown that the number of random lines which meet a given closed convex surface is measured by the area of the surface. It is an interesting question to determine this measure when the surface is not convex.

The measure of the number of *random planes* which meet a given convex surface is shown to depend upon the integral

$$\iint p^{\sin\theta\, d\theta\, d\phi},$$

where p is the perpendicular from any fixed pole upon the tangent plane, and θ, ϕ its angular coordinates. It remains to discover a geometrical representation for this integral.

A short discussion followed upon this paper, in which Messrs. S. Roberts, A. Smith, Cayley, and the President took part.

The Chairman then read a paper "On a Law of Periodicity in the derivation of point from point on Cubic Curves, and its connection with the Theory of Allineation."

The following presents were made to the Society:—

"On Centres, Faisceaux, and Envelopes of Homology;" and "A Note on Prof. Tait's 'Quaternion Path' to Determinants of the Third Order," by the Rev. Hugh Martin, M.A.;

"A Note on the importance of the Spectroscopical Examinations of the vicinity of the Sun when totally eclipsed," &c., by Prof. Brayley, F.R.S.;

And "Un Teorema intorno alle forme quadratiche non omogenee fra due variabili," by Prof. Cremona.

January 23rd, 1868.

Prof. HIRST, Treasurer, in the Chair.

Mr. H. Perigal, F.R.A.S., was elected a Member.

A vote of thanks was carried unanimously to Herr Borchardt for his present of *Crelle's Journal.*

Prof. Clerk Maxwell then proceeded to make the following remarks upon "The Construction of Stereograms of Surfaces":—

"To make a surface visible, lines must be drawn upon it; and to exhibit the nature of the surface, these lines ought to be traced on the surface according to some principle, as for instance the contour lines and lines of greatest slope on a surface may be drawn. Monge represents surfaces to the eye by their two systems of lines of curvature, which have the advantage of being independent of the direction assumed for the coordinate axes. For stereoscopic representation it is necessary to choose curves which are easily followed by the eye, and which are sufficiently different in form to prevent a curve of the one figure from being visually united with any other than the corresponding curve in the other figure. I have found the best way in practice to be as follows:—First determine how many curves are to be drawn on the surface, and at what intervals, and find the numerical values of the coordinates of points on these curves. A convenient method of drawing the figures according to Cartesian coordinates is to draw an equilateral triangle, find its centre of gravity, and take a point about $\frac{1}{30}$ of the side of the triangle distant horizontally from the centre of gravity as the origin,

and the lines joining this point with the angles as unit axes. For the other figure, the point must be taken on the opposite side of the centre of gravity. I have found that this rule gives a convenient amount of relief to the figures. When the coordinates of a point are easily expressed in terms of tetrahedral coordinates x, y, z, w, I draw the two lines $(x = 0,\ y = 0)$ and $(z = 0, w = 0)$ in both figures. By means of a sector I divide the first line in P in the ratio of z to w, and the second in Q in the ratio of x to y; I then lay a rule along the line PQ (without drawing the line) and divide PQ in the ratio of $x+y$ to $z+w$, in order to find the point R in the figure. By drawing the two lines once for all in each figure, and performing the same process of finding ratios in each figure, we get each point without making any marks on the paper, which come to be very troublesome in complicated figures. In this way I have drawn the figures of cyclides, &c."

A vote of thanks was passed to Mr. Maxwell; and the Chairman having pointed out the educational value of these stereograms, Mr. Maxwell stated that he had prepared most of the specimens exhibited for publication. The members present were enabled, after the meeting, to examine a large number of stereograms by means of a Real Image Stereoscope constructed after Mr. Maxwell's directions.

There were on view stereograms of the lines of curvature of the ellipsoid, and its surface of centres; of the wave surface of Fresnel, of confocal spherical ellipses, of concyclic spherical ellipses, showing the form of Laplace's coefficient of the second order, of twisted cubics, of Gordian knots of the form

$$x = a \sin pt, \quad y = b \sin qt, \quad z = c \cos rt,$$

and of 4 forms of the cyclide.

The following paper was then read:—

On Reciprocal Diagrams in Space, and their relation to Airy's Function of Stress. *By* J. Clerk Maxwell, M.A., F.R.SS.L. and E.

Let F be any function of the coordinates x, y, z of a point in space, and let ξ, η, ζ be another system of coordinates which we may suppose referred to axes parallel to the axes of x, y, z, but at such a distance (in thought) that figures referred to x, y, z do not interfere with the figures referred to ξ, η, ζ.

We shall call the figure or figures referred to x, y, z the First Diagram, and those referred to ξ, η, ζ the Second Diagram.

Let the connexion between the two diagrams be expressed thus—

$$\xi = \frac{dF}{dx}, \qquad \eta = \frac{dF}{dy}, \qquad \zeta = \frac{dF}{dz}.$$

When the form of F is known, ξ, η and ζ may be found for every value of x, y, z, and the form of the second diagram fixed.

To complete the second diagram, let a function ϕ of ξ, η, ζ be found from the equation

$$\phi = x\xi + y\eta + z\zeta - \mathrm{F};$$

then it is easily shown that

$$x = \frac{d\phi}{d\xi}, \qquad y = \frac{d\phi}{d\eta}, \qquad z = \frac{d\phi}{d\zeta}.$$

Hence the first diagram is determined from the second by the same process that the second is determined from the first. They are therefore Reciprocal Diagrams both as regards their form and their functions.

But reciprocal diagrams have a mechanical significance which is capable of extensive applications, from the most elementary graphic methods for calculating the stresses of a roof to the most intricate questions about the internal molecular forces in solid bodies. I shall indicate two independent methods of representing internal stress by means of reciprocal diagrams.

First Method. Let a, b be any two contiguous points in the first diagram, and α, β the corresponding points in the second. Let an element of area be described about ab perpendicular to it. Then if the stress per unit of area on this surface is compounded of a *tension* parallel to $\alpha\beta$ and equal to $\mathrm{P}\frac{\alpha\beta}{ab}$ and a *pressure* parallel to ab and equal to $\mathrm{P}\left(\frac{d^2\mathrm{F}}{dx^2} + \frac{d^2\mathrm{F}}{dy^2} + \frac{d^2\mathrm{F}}{dz^2}\right)$, P being a constant introduced for the sake of homogeneity, then a state of internal stress defined in this way will keep every point of the first figure in equilibrium.

The components of stress, as thus defined, will be

$$p_{xx} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dx^2} - \Delta^2\mathrm{F}\right), \quad p_{yy} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dy^2} - \Delta^2\mathrm{F}\right), \quad p_{zz} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dz^2} - \Delta^2\mathrm{F}\right)$$

$$p_{yz} = \mathrm{P}\frac{d^2\mathrm{F}}{dy\,dz}, \qquad p_{zx} = \mathrm{P}\frac{d^2\mathrm{F}}{dz\,dx}, \qquad p_{xy} = \mathrm{P}\frac{d^2\mathrm{F}}{dx\,dy},$$

and these are easily shown to fulfil the conditions of equilibrium.

If any number of states of stress can be represented in this way, they can be combined by adding the values of their functions (F), since the quantities are linear. This method, however, is applicable only to certain states of stress; but if we write

$$p_{xx} = \frac{d^2\mathrm{B}}{dz^2} + \frac{d^2\mathrm{C}}{dy^2}, \quad p_{yy} = \frac{d^2\mathrm{C}}{dx^2} + \frac{d^2\mathrm{A}}{dz^2}, \quad p_{zz} = \frac{d^2\mathrm{A}}{dy^2} + \frac{d^2\mathrm{B}}{dx^2},$$

$$p_{yz} = -\frac{d^2\mathrm{A}}{dy\,dz}, \qquad p_{zx} = -\frac{d^2\mathrm{B}}{dz\,dx}, \qquad p_{xy} = -\frac{d^2\mathrm{C}}{dx\,dy},$$

we get a general method at the expense of using three functions A, B, C, and of giving up the diagram of stress.

Second Method. Let a be any element of area in the first diagram,

and a the corresponding area in the second. Let a uniform normal pressure equal to P per unit of area act on the area a, and let a force equal and parallel to the resultant of this pressure act on the area a, then a state of internal stress in the first figure defined in this way will keep every point of it in equilibrium.

The components of stress as thus defined will be

$$p_{xx} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dy^2}\frac{d^2\mathrm{F}}{dz^2} - \left|\frac{d^2\mathrm{F}}{dy\,dz}\right|^2\right), \qquad p_{yy} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dz^2}\frac{d^2\mathrm{F}}{dx^2} - \left|\frac{d^2\mathrm{F}}{dz\,dx}\right|^2\right),$$

$$p_{zz} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dx^2}\frac{d^2\mathrm{F}}{dy^2} - \frac{d^2\mathrm{F}}{dx\,dy}\right),$$

$$p_{yz} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dz\,dx}\frac{d^2\mathrm{F}}{dx\,dy} - \frac{d^2\mathrm{F}}{dx^2}\frac{d^2\mathrm{F}}{dy\,dz}\right), \quad p_{zx} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dx\,dy}\frac{d^2\mathrm{F}}{dy\,dz} - \frac{d^2\mathrm{F}}{dy^2}\frac{d^2\mathrm{F}}{dz\,dx}\right)$$

$$p_{xy} = \mathrm{P}\left(\frac{d^2\mathrm{F}}{dy\,dz}\frac{d^2\mathrm{F}}{dz\,dx} - \frac{d^2\mathrm{F}}{dz^2}\frac{d^2\mathrm{F}}{dx\,dy}\right).$$

This is a more complete, though not a perfectly general, representation of a state of stress; but as the functions are not linear, it is difficult of application.

If however we confine ourselves to problems in two dimensions, either method leads to the expression of the three components of stress in terms of the function introduced by the Astronomer Royal.*

$$p_{xx} = \mathrm{P}\frac{d^2\mathrm{F}}{dy^2}, \qquad p_{xy} = -\mathrm{P}\frac{d^2\mathrm{F}}{dx\,dy}, \qquad p_{yy} = \mathrm{P}\frac{d^2\mathrm{F}}{dx^2}.$$

Here $\xi = \dfrac{d\mathrm{F}}{dx}, \quad y = \dfrac{d\mathrm{F}}{dy}, \quad \mathrm{F} + \phi = x\xi + y\eta,$

and if ab and $a\beta$ be corresponding lines, the whole stress across the line ab is perpendicular to $a\beta$ and equal to $\mathrm{P}a\beta$.

The applications of this method in two dimensions are too numerous to be considered here.

Mr. J. J. Walker read a paper "On the Anharmonic-Ratio Sextic." The author proposed to himself in this paper the formation of the sextic which gives the six values of the anharmonic ratio of 4 points on a line. The subject has been treated by M. Painvin, "Equation des rapports anharmoniques," (Nouvelles Annales, t. 19, 1860,)—but Mr. Walker claims to have arrived at his result by a simpler method, and to have put it in a more elegant form. In the course of discussion on this paper, Mr. Maxwell asked if any member present could point out a method of determining in what cases all the possible parts of the impossible roots of an equation are negative. In studying the motion of certain governors for regulating machinery, he had found that the stability of the motion depended on this condition, which is easily obtained for a cubic, but becomes more difficult in the higher degrees.† Mr. W. K. Clifford said that, by forming an equation whose roots are the

* On Strains in the Interior of Beams. Phil. Trans., 1863.

† On Governors. Proceedings of the Royal Society, March 5, 1868.

sums of the roots of the original equation taken in pairs and determining the condition of the real roots of this equation being negative, we should obtain the condition required.

The following presents were made to the Society's library:—

"Preliminari di una Teoria Geometrica delle Superficie," parts 1, 2, 1866; and "Sulle trasformzioni geometriche delle figure piane," 2nd Note, 1865. By Prof. Clausius.

"Abhandlungen über die Mechanische Wärmetheorie von R. Clausius." By Prof. Hirst.

"On the Cyclide," and "On the Equilibrium of a Spherical Envelope." By Prof. Clerk Maxwell.

Also the "Journal of the Institute of Actuaries," January, 1868.

February 27th, 1868.

Prof. SYLVESTER, President, in the Chair.

Mr. J. Glaisher, F.R.S., F.R.A.S., Royal Observatory, Greenwich, was elected a Member; and Mr. H. Perigal, F.R.A.S., was admitted into the Society by the President.

Mr. Spottiswoode, V.P., then read a paper "On the Equilibrium of Forces in Space;" and Mr. W. K. Clifford, B.A., a paper "On Circles and Spheres."

The following presents were made to the Society:—Hargreave's "Essay on the Resolution of Algebraic Equations," from Mrs. Hargreave, through the Rev. Dr. Salmon; and a "Note sur l'équilibre des forces dans l'espace," from Mr. Spottiswoode, the author.

March 26th, 1868.

Prof. SYLVESTER, President, in the Chair.

Mr. J. F. Moulton, B.A., St. John's College, Cambridge, was elected a Member.

Prof. H. J. Stephen Smith, V.P., read a paper on "The construction of the last point of intersection of a Cubic Curve by a curve of superior order," and "A Geometrical Note on the Concomitants of a Binary Cubic."

Prof. Cayley made some remarks on a mode of generation of a sibi-reciprocal surface, that is, a surface the reciprocal of which is of the same order and has the same singularities as the original surface.

If a surface be considered as the envelope of a *plane* varying according to given conditions, this is a mode of generation which is essentially not sibi-reciprocal; the reciprocal surface is given as the locus of a

point varying according to the reciprocal conditions. But if a surface be considered as the envelope of a *quadric surface* varying according to given conditions, then the reciprocal surface is given as the envelope of a *quadric surface* varying according to the reciprocal conditions; and if the conditions be sibi-reciprocal, it follows that the surface is a sibi-reciprocal surface. For instance, considering the surface which is the envelope of a quadric surface touching each of 8 given lines; the reciprocal surface is here the envelope of a quadric surface touching each of 8 given lines; that is, the surface is sibi-reciprocal. So again, when a quadric surface is subjected to the condition that 4 given points shall be in regard thereto a conjugate system, this is equivalent to the condition that 4 given planes shall be in regard thereto a conjugate system—or the condition is sibi-reciprocal; analytically the quadric surface $ax^2+by^2+cz^2+dw^2=0$ is a quadric surface subjected to a sibi-reciprocal system of six conditions. Impose on the quadric surface two more sibi-reciprocal conditions,—for instance, that it shall pass through a given point and touch a given plane,—the envelope of the quadric will be a sibi-reciprocal surface. It was noticed that in this case the envelope was a surface of the order (=class) 12, and having (besides other singularities) the singularities of a conical point with a tangent cone of the class 3, and of a curve of plane contact of the order 3. In the foregoing instances the number of conditions imposed upon the quadric surface is 8; but it may be 7, or even a smaller number. An instance was given of the case of 7 conditions, viz.,—the quadric surface is taken to be $ax^2+by^2+cz^2+dw^2=0$ (6 conditions) with a relation of the form

$$Abc+Bca+Cab+Fad+Gbd+Hcd=0$$

between the coefficients (1 condition); this last condition is at once seen to be sibi-reciprocal; and the envelope is consequently a sibi-reciprocal surface—viz., it is a surface of the order (=class) 4, with 16 conical points and 16 conics of plane contact. It is the surface called by Prof. Cayley the "tetrahedroid," (see his paper "Sur la surface des ondes," Liouv. tom. xi. (1846) pp. 291—296), being in fact a homographic transformation of Fresnel's Wave Surface.

[Prof. Cayley adds an observation which has since occurred to him. If the quadric surface $ax^2+by^2+cz^2+dw^2=0$, be subjected to touch a given line, this imposes on the coefficients a, b, c, d, a relation of the above form, viz., the relation is

$$A^2bc+B^2ca+C^2ab+F^2ad+G^2bd+H^2ad=0;$$

where A, B, C, F, G, H are the "six coordinates" of the given line, and satisfy therefore the relation $AF+BG+CH=0$. It is easy to see that there are 8 lines for which the squared coordinates have the same values A^2, B^2, C^2, F^2, G^2, H^2; these 8 lines are symmetrically situate in regard to the tetrahedron of coordinates, and moreover they lie in a hyperboloid. The quadric surface, instead of being defined as above,

may, it is clear, be defined by the equivalent conditions of touching each of the 8 given lines: that is, we have the envelope of a quadric surface touching each of 8 given lines; these lines not being arbitrary lines, but being a system of a very special form. By what precedes, the envelope is a quartic surface. It appears, however, that in virtue of the relation $AF+BG+CH=0$, this is no longer a proper quartic surface, but that it resolves itself into the above-mentioned hyperboloid taken twice. That is, restoring the original A, B, &c., in place of A^2, B^2, &c., the envelope of the quadric $ax^2+by^2+cz^2+dw^2=0$, where a, b, c, d vary, subject to the condition $Abc+Bca+Cab+Fad+Gbd+Hcd=0$, (which is in general a tetrahedroid), is when A, B, C, F, G, H are the squared coordinates of a line (or, what is the same thing, when $\sqrt{AF}+\sqrt{BG}+\sqrt{CH}=0$) a hyperboloid taken twice, viz., this is the hyperboloid passing through the given line and through the symmetrically situate seven other lines.]

The President then made a statement on a property of *Residuals*.

Prof. Hirst next presented the following paper,—

On the Inscription, by a Simplification of Sir W. R. Hamilton's Process of Reduction, of closed n*-gons in any quadric, so that the sides of each shall pass in order through* n *given points.* By Martin Gardiner, C.E.

Let S be the given quadric, and $o_1, o_2, \ldots o_n$ the given points, and let $a_1a_2 \ldots a_na_{n+1}$ be any inscribed n-gon, open or closed, whose sides $a_1a_2, a_2a_3, \ldots a_n a_{n+1}$ pass in order through the respective points $o_1, o_2, \ldots o_n$. (See *Note* 1° at end of paper.)

Let P be any plane in space, chosen at pleasure, not tangent to S.

Now if p_1 be the point in which the line o_1o_2 pierces P, we know that another point q_1 can be found in this line, such that p_1a_1 and q_1a_3 will intersect in some point b_2 in the surface S. Again, if p_2 be the point in which q_1o_3 pierces P, we can find another point q_2 in the same line, such that p_2b_2 and q_2a_4 will intersect in some point b_3 in the surface S. And it is evident we may thus proceed until we have an inscribed n-gon $a_1b_2b_3 \ldots b_na_{n+1}$ whose first $n-1$ sides pass through $n-1$ known points $p_1, p_2, \ldots p_{n-1}$ in one plane P, and whose last side b_na_{n+1} passes through a known point ρ_n.

It is also evident that if we draw any straight line L in the plane P, we can, in a similar manner, find $n-2$ points $r_1, r_2, \ldots r_{n-2}$ in this line, and a $(n-1)$th point ρ_{n-1} not necessarily in it, such that any open or closed n-gon whose sides pass in order through the n given points will have its extremities coincident with those of an inscribable n-gon whose sides pass in order through the n points $r_1, r_2, \ldots r_{n-2}, \rho_{n-1}, \rho_n$.

Let $c_1c_2 \ldots . c_n c_1$ be an inscribed closed n-gon whose sides pass in order through the n points $r_1, r_2, \ldots . r_{n-2}, \rho_{n-1}, \rho_n$.

If n be odd, $n-2$ must be odd, and we know that the chord $c_{n-1}c_1$ must pass through a fixed known point β in the line L; and we at once perceive $c_1c_{n-1}c_nc_1$ is an inscribed closed 3-gon whose sides pass in order through the three points β, ρ_{n-1}, ρ_n. But if in the line $\rho_n \rho_{n-1}$ we assume the point β' conjugate to β, we can find another point p in the same line such that $\beta' c_{n-1}$ and pc_1 will intersect in some point in the surface. Now β and β' being conjugates, the chord pc_1 which passes through the point p must be in contact with the line $\beta\beta'$ and its reciprocal, and is therefore known. Hence we know the two points c_1 and c_1' in which pc_1 pierces the quadric—which points must be first extremities of inscribable closed n-gons whose sides pass in order through the n given points $o_1, o_2, \ldots o_n$.

We may observe that when β, ρ_{n-1}, ρ_n are not collinear, or do not form a self-conjugate triad, then p will not be situated either in the line $\beta\beta'$ or its reciprocal, and there can be but one straight line pc_1', and therefore two, and but two, first extremities c_1 and c_1' both real or both imaginary as may be.

When β, ρ_{n-1}, ρ_n are in one straight line, then β, β', p are in the same line, and it is obvious that the first extremities of the inscribable closed n-gons have as locus the trace of the plane containing the point p and the reciprocal of this line. When the points β, ρ_{n-1}, ρ_n form a self-conjugate triad, then β' and p are coincident with ρ_{n-1} and ρ_n, and it is obvious that the first extremities of the inscribable closed n-gons have as locus the trace of the plane $\beta\, \rho_{n-1}\, \rho_n$. Moreover, it is evident that if there be a quadric and any odd number n of points in a plane such that the locus of the first extremities of the inscribable closed n-gons is the trace of the plane, then we can find a self-conjugate triad of points in the plane such that the extremities of each inscribable closed or open n-gon will be coincident with the extremities of an inscribable 3-gon whose sides pass in order through the points of the triad. And from this we know that the closing chords of all inscribable open n-gons will, in such case, pass through the pole of the plane containing the points.

If n be even, then, if we assume any point β in L, we can find another point γ in the same line such that βc_1 and γc_{n-1} will intersect in some point k in the surface S. But if xx be the straight line which pierces in the first extremities of closed 3-gons whose sides pass in order through the three points γ, ρ_{n-1}, ρ_n we know, from the theory of plane conics, that, if anywhere in xx we assume a point β', we can find another point δ in the same line such that every point in the trace of the plane $\gamma\, \rho_{n-1}\, \rho_n$ will be the first extremity of an inscribable closed 5-gon whose sides pass in order through the 5 points γ, ρ_{n-1}, ρ_n, β', δ; and therefore we know that the closing chords of all inscribable 5-gons,

whose sides pass through these 5 points, will pass through the point δ' which is the pole of the plane containing them. Hence also we perceive that c_1 is the first extremity of an inscribable closed 4-gon whose sides pass in order through the four points β, β', δ, δ', the last two of which are conjugates; but we can choose β' in xx conjugate to β; and therefore, by so doing, it is evident each of the points of puncture of a straight line drawn so as to intersect the lines $\beta\beta'$, $\delta\delta'$ and their reciprocals, will be the first extremity of an inscribable closed 4-gon whose sides pass in order through the four points β, β' δ, δ', and will also be the first extremity c_1 of an inscribable closed n-gon whose sides pass in order through the n points o_1, o_2, . . . o_n. Now, we know that two straight lines yy, $y'y'$ can (generally) be drawn to cut the lines $\beta\beta'$, $\delta\delta'$ and their reciprocals, and that yy and $y'y'$ must be reciprocals.

If the lines $\beta\beta'$, $\delta\delta'$ form one straight line, it is evident their reciprocals will be in one straight line, and that every point in the surface will be the first extremity of an inscribable closed n-gon. If the lines $\beta\beta'$, $\delta\delta'$ be reciprocals, then also it is evident that every point in the surface will be a first extremity of an inscribable closed n-gon.

Moreover, from this case (of n even), it is evident that if n be odd, then will the reciprocal of the straight line which pierces in first extremities of the inscribable closed n-gons, pierce in points (real or imaginary as may be) which are first extremities of inscribable closed $2n$-gons, whose first n sides and whose second n sides pass in order through the n given points.

Note 1°.—A previous knowledge of the following elementary theorems, and of their respective converses will enable the reader to follow more readily the preceding method of investigation:—

1. If a_1a_2, a_2a_3 be two variable chords of a quadric passing through two fixed conjugate points o_1, o_2, then will the variable chord a_1a_3 intersect in every position the line o_1o_2, and its reciprocal with respect to the surface.

2. If o_1, o_2, o_3, o_4, be a self-conjugate quartet of points in respect to any quadric, then will every point in the surface be the first extremity of an inscribable closed 4-gon whose sides pass through the points.—The proof of this is evident from the preceding theorem.

3. If in a quadric there be inscribed any closed n-gon of an even number n of sides, $n-1$ of which pass through $n-1$ fixed points in one straight line, then will the remaining side pass through a fixed determinable point in the same straight line.

Note 2°.—After reducing the problem to the inscription of closed n-gons whose first $n-1$ sides pass through $n-1$ points all in one plane cutting the quadric, we might easily solve it by drawing the line yy, in

that plane, which contains the first extremities of closed $(n-1)$-gons, &c. &c.

Note 3°.—In the case in which n is even we might with advantage simplify the solution by a convenient choice of the plane P and line L. Thus:—Suppose the plane P to be any central plane of the quadric, and that the line L is any diameter in this plane. Then we could choose the point β coincident with the centre of the quadric, and find the corresponding point γ in L. We know that the point β' conjugate to β would be at infinity in the line xx, &c., the line $\beta\beta'$ being a known diametral line parallel to xx; and we perceive that any straight line which cuts $\beta\beta'$ and its reciprocal must be parallel to a known diametral plane B which is conjugate to the line $\beta\beta'$, &c. Hence the finding of first extremities of closed n-gons would be reduced to the drawing of lines parallel to the plane B each of which would be in contact with $\beta\beta'$, $\delta\delta'$ and the reciprocal of $\delta\delta'$.

Now such lines could be easily found. Let h, i be the points in which $\delta\delta'$ and its reciprocal pierce B; and let hg, ij be their projections on B, in which the projecting ordinates are parallel to $\beta\beta'$. Then evidently the projection of either of the sought lines would pass through the centre β and cut off on hg and ij intercepts from h and i having to each other a known ratio; and (by the "section of ratio" of Apollonius) we could determinate these projections, and therefore also the sought lines.

N.B.—It is evident (see pages 548, 549, Vol. 4 of "The Proceedings of the Royal Irish Academy"), that Sir William R. Hamilton's method of reduction would be much simplified by similar projections.

Newcastle, New South Wales
Nov. 19, 1867.

Mr. A. De Morgan, V.P., made some remarks on the application of the signs *plus* and *minus* to the distribution of terms in logic.

The following presents were made to the Society:—

"Three Supplements to Sir W. Rowan Hamilton's Essay on the Theory of Systems of Rays;" from Dr. Ingleby.

"Besant's Hydrostatics and Hydrodynamics;" and "Rapport fait à l'Académie royale des Sciences des Pays Bas;" from Dr. Hirst.

[The paper on "The Equilibrium of Forces in Space," read at the February Meeting, was withdrawn by the author, in consequence of many of the results having been anticipated in a paper "On the Six Coordinates of a Line," by Prof. Cayley, previously presented to, and subsequently published by, the Cambridge Philosophical Society. In reference to that paper, § 54, it may be remarked that the equation of

the hyperboloid passing through three given lines is at once seen to be

$$\begin{vmatrix} X, & Y, & Z \\ X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \end{vmatrix} = 0;$$

where $X = bz - cy + f$, $X_1 = \ldots\ldots$, $X_2 = \ldots\ldots$,
$Y = cx - az + g$, $Y_1 = \ldots\ldots$, $Y_2 = \ldots\ldots$,
$Z = ay - bx + h$, $Z_1 = \ldots\ldots$, $Z_2 = \ldots\ldots$,

and a, b, c, f, g, h; $a_1, b_1 \ldots\ldots$; $a_2, b_2 \ldots\ldots$ are the six coordinates of the three lines respectively.

Also, in reference to §§ 45 and 50, the symmetrical expressions, there mentioned as desiderata, will be found in a paper by the author, "Note sur l'équilibre des forces dans l'espace," Comptes Rendus, Jan. 13, 1868.

N.B. In this Note, and Mr. Spottiswoode's last-mentioned Paper, the quantities a, b, c, correspond to Prof. Cayley's f, g, h, and *vice versâ*.]

April 23rd, 1868.

Prof. SYLVESTER, President, in the Chair.

Mr. J. T. Graves, M.A., F.R.S., and Dr. O. Henrici, M.F.E., were elected Members; and Messrs. J. Glaisher, F.R.S., and H. MacNeile, B.A., were admitted into the Society.

Mr. T. Cotterill read a paper "On the 8 points of intersection of 3 Quadric Surfaces." Prof. Hirst and Messrs. Clerk-Maxwell and Clifford took part in a discussion upon the paper. (This paper has since been withdrawn, as the author proposes to bring a more extended paper upon the subject before the Society.)

The following presents were made to the Society:—

"Catalogue of Scientific Papers, vol. i., A to CLU;" from the Royal Society.

"Annali di Matematica," 3 parts; from Prof. Cremona.

"Comptes Rendus. Rapport sur un mémoire de M. Mannheim sur le déplacement d'une figure de forme invariable. (Rapporteur, M. Chasles);" from Prof. Mannheim.

May 28th, 1868.

Prof. SYLVESTER, President, in the Chair.

The Rev. P. T. Main, M.A., and Mr. W. H. H. Hudson, M.A., Fellow of St. John's College, Cambridge, were elected Members. Dr. O. Henrici was admitted into the Society by the President.

Prof. H. J. Stephen Smith then read a paper "On the Theory of

certain Systems of Conics which present themselves in connexion with Cubic Curves."

The President communicated to the Society the following Solution of Lexell's Problem, by the Lord Bishop of Limerick :—

Since $A+B+C-\pi =$ the area of the spherical triangle, construct on the given base AB an isosceles triangle AOB, whose base angles are each equal to $\frac{1}{2}$ (area $+\pi$). With the vertex O of this triangle as centre, and for radius an arc supplemental to one of the equal sides of the isosceles triangle, describe a small circle of the sphere. This will be the locus sought.

The President, in continuation, remarked upon the riches of Spherical as contrasted with Plane Trigonometry.

Mr. Crofton made the following remarks on the double Integral,

$$\iint \frac{\sin\theta}{\mathrm{T}\mathrm{T}'}\,dx\,dy;$$

where T, T′ are the tangents drawn from (x, y) to a given closed convex contour, and θ is the angle they contain.

If an infinite number of tangents to the given contour be drawn, each tangent making a constant infinitesimal angle (δ) with the preceding one, the intersections of these tangents with one another compose an infinite assemblage of points covering the plane; and it may be proved that the *density* of these points is $\frac{1}{\delta^2}\frac{\sin\theta}{\mathrm{T}\,\mathrm{T}'}$;

but as the whole number of tangents is $\frac{2\pi}{\delta}$, the number of their intersections is $\frac{2\pi^2}{\delta^2}$. Hence the value of the integral for the whole surface of the plane, exterior to the given contour, which may be of any form, provided it contain no *salient points*, will be

$$\iint \frac{\sin\theta}{\mathrm{T}\,\mathrm{T}'}\,d\mathrm{S} = 2\pi^2.$$

If the integral extend only over the *annulus* between the proposed contour and an exterior line along which θ has a constant value (α),

$$\iint \frac{\sin\theta}{\mathrm{T}\,\mathrm{T}'}\,d\mathrm{S} = 2\pi\,(\pi-\alpha).$$

If the contour contain salient points, then for any such point where the boundary changes direction abruptly through an angle A, a certain number $\left[\text{viz., } \frac{1}{2}\left(\frac{\mathrm{A}}{\delta}\right)^2\right]$ of the intersections coincide at the salient point. This number must of course be subtracted from the above integrals; hence, if the contour contain any number of salient points, the first integral will be

$$\iint \frac{\sin\theta}{\mathrm{T}\,\mathrm{T}'}\,d\mathrm{S} = 2\pi^2 - \tfrac{1}{2}\Sigma\mathrm{A}^2;$$

likewise for the second. Thus, for a regular polygon the value is $2\pi^2\left(1-\frac{1}{n}\right)$. For a finite straight line the value is π^2, instead of $2\pi^2$; a result which may be verified, the tangents T, T′ becoming here the vectors r, r' to the extremities of the line.

If, instead of the above system of tangents, altering by constant angles, we take an infinity of equidistant points ranged along the given contour, and draw tangents at these points, we shall find that the density of the intersections of these tangents is proportional to

$$\frac{\rho\rho'}{\mathrm{TT}'}\sin\theta,$$

where ρ, ρ' *are the radii of curvature of the contour at the points of contact of the tangents* T, T′; hence we obtain the integral

$$\iint\frac{\rho\rho'}{\mathrm{TT}'}\sin\theta d\mathrm{S}=\tfrac{1}{2}\mathrm{L}^2,$$

extended over the whole outer surface; where L is the perimeter of the contour.

June 25th, 1868.

Prof. SYLVESTER, President, in the Chair.

Prince Camille de Polignac was elected a Member, and admitted into the Society by the President, as also were Messrs. W. J. C. Miller, J. M. Wilson, and W. S. B. Woolhouse.

A paper was read by Mr. J. M. Wilson on "Euclid as a Text-book of Elementary Geometry." Prof. Hirst, Messrs. De Morgan, Ellis, Hemming, and A. Smith took part in the discussion upon this paper, and Mr. Wilson replied. [The paper has since been printed *in extenso* in the *Educational Times* for September, 1868, and allusion is made to it in the *Athenæum*, July 18th, 1868, in a review of Mr. Wilson's Geometry.]

Prince Camille de Polignac then read the following paper:—

On a Problem in Combinations.

I. m letters $abc\ldots a'b'c'\ldots khl\ldots$ being given, it is proposed to find the number of combinations which can be formed with these letters, n and n together, with the condition that a and a' must not enter the same combination, nor b and b', c and c' &c.

The letters $khl\ldots$, which are submitted to no restriction, we shall call *free* letters, the others *associative* letters; a and a', b and b', &c., will be called *associated* or *corresponding* letters.

The combinations themselves we shall call *restricted combinations*, and their number will be represented by the notation ${}^{q}\mathrm{K}_{m,n}$ in which

m indicates the total number of letters, n how many enter each combination, and q the number of free letters. Thus $m-q$ indicates the number of *associative* letters, and it is plain that $m-q$ must be an even number.

To begin with the simplest case, let us first suppose $q = 0$, whence $m = 2p$. Supposing all the restricted combinations to have been written out, let us form in each all possible permutations of letters. We shall thus obtain the *restricted permutations* of $2p$ letters, n and n together, and if we designate their number by $M_{2p,n}$ we shall have

$$K_{2p,n} = \frac{M_{2p,n}}{\Pi_n},$$

where Π_n stands for the number of ordinary permutations of n letters, equal to $1.2.3.....n$.

In order to define $M_{2p,n}$ let us throw together all the permutations *beginning* with the same letter. We shall thus form $2p$ groups, in which the first letters will be respectively, $a, b, c \ldots a', b', c' \ldots\ldots$, and it is obvious, from symmetry, that each group will be composed of the same number of permutations.

But if we strike off a from the first group, we shall obtain restricted permutations of $(n-2)$ letters, as a' does not enter that group. The number of permutations in the first group is therefore $M_{2p-2,n-1}$, and consequently

$$M_{2p,n} = 2pM_{2p-2,n-1};$$

whence

$$M_{2p-2,n-1} = (2p-2)\, M_{2p-4,n-2}$$
$$M_{2p-4,n-2} = (2p-4)\, M_{2p-6,n-4}$$
$$\vdots \qquad\qquad \vdots$$
$$M_{2p-2n+4,2} = (2p-4)\, M_{2p-2n+2,1}.$$

If we multiply together these equations, and observe that $M_{2p-2n+2} = 2p-2n+2$, we find

$$M_{2p,n} = 2p\,(2p-2)\,(2p-4).....(2p-2n+2);$$

hence

$$K_{2p,n} = \frac{2p\,(2p-2)\,\ldots\ldots\,(2p-2n+2)}{1.2.3\,\ldots\ldots\,n} = 2^n C_{p,n},$$

$C_{p,n}$ indicating the number of ordinary combinations of n letters out of p.

When $n = p$, $\qquad K_{2p,p} = 2^p$.

II. Let us now pass on to the general case in which there are q free letters and $m = 2p+q$.

Any specific combination out of the system whose number we have agreed to denote by ${}^qK_{2p+q,n}$ (and which, for brevity, we shall in future call a combination *appurtenant* to ${}^qK_{2p,q,n}$) will contain α free letters and β associative letters.

If from that combination we strike off the free letters, the remaining letters will give one restricted combination appurtenant to $K_{2p,\beta}$.

But if we strike off the associative letters, we shall obtain one ordinary combination appurtenant to $C_{q,\alpha}$.

Hence the number of restricted combinations in which occur α free, and β associative, letters is $C_{q,\alpha} \times K_{2p,\beta}$.

But we have obviously ${}^{q}K_{2p+q,n} = \Sigma C_{q,\alpha} \cdot K_{2p,\beta}$, where α and β can range from 0 to n, with the condition $\alpha+\beta=n$; hence

$$\begin{aligned} {}^{q}K_{2p+q,n} = K_{2p,n} &+ C_{q,1} \cdot K_{2p,n-1} + C_{q,2} \cdot K_{2p,n-2} + \ldots\ldots \\ &+ C_{q,n-2} \cdot K_{2p,2} + C_{q,n-1} \cdot K_{2p,1} + C_{q,n}, \end{aligned}$$

and we know how to form $K_{2p,n}$, $K_{2p,n-2}$, &c.

III. On a determinant involving $K_{2p,n}$.

We shall now give a new expression for $K_{2p,n}$ based upon a method entirely different from the first, and which consists in striking off from the total number of ordinary combinations $C_{2p,n}$ such as contain one or more couples of corresponding letters.

Let Q_h represent the aggregate combinations which contain h letters and their associates; we shall have $K_{2p,n} = C_{2p,n} - \Sigma Q_h$, where h ranges from 1 to $\frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

But Q_h is composed of combinations such as $abc\ldots\ldots a'b'c' \cdot P$, where P represents a product of $n-2h$ letters cleared of corresponding letters. P can therefore take $K_{2p\ 2h,n-2h}$ values.

On the other hand, the factor $abc\ldots\ldots a'b'c'\ldots$ can be written $\alpha\beta\gamma\ldots$, where every Greek letter stands for the product of a Roman letter and its associate; and, in this form, it is clear that $abc\ldots\ldots a'b'c'\ldots$ can take $C_{p,h}$ values; hence $Q_h = C_{p,h} \cdot K_{2p-2h,n\ 2h}$; whence finally

$$K_{2p,n} = C_{2p,n} - \Sigma C_{p,h} \cdot K_{2p-2h,n-2h}.$$

If n be even, replacing it by $2n$, we find

$$\begin{aligned} (1)\ \ldots\ldots\ K_{2p,2n} = C_{2p,2n} - \{ & C_{p,1} \cdot K_{2p-2,2n-2} + C_{p,2} \cdot K_{2p-4,2n-4} + \ldots \\ & \ldots + C_{p,n-1} \cdot K_{2p-2n+2,2} + C_{p,n} \}. \end{aligned}$$

Let us write, generally,*

$$X_n = K_{2p,2n}, \qquad Q_n = C_{p,n} - C_{2p,2n}.$$

If in (1) we change successively p and n into $p-1$, $n-1$; $p-2$, $n-2$, &c., we shall form the n following equations.

$$\begin{aligned} C_{p,n-1}X_1 + C_{p,n-2}X_2 + \ldots\ldots C_{p,1}X_{n-1} + X_n + Q_n &= 0 \\ C_{p-1,n-2}X_1 + C_{p-1,n-3}X_2 + \ldots\ldots C_{p-1,1}X_{n-2} + X_{n-1} + Q_{n-1} &= 0 \\ \vdots \qquad\qquad \vdots & \\ C_{p-n+2,1}X_1 + X_2 + Q_2 &= 0 \\ X_1 + Q_1 &= 0 \end{aligned}$$

* When in these equations n is replaced by $n-h$, p must also be replaced by $p-h$. Thus $X_{n-1} = K_{2p-2,2n-2}$, $Q_{n-1} = C_{p-1,n-1} - C_{2p-2\ 2n-2}$, &c.

and the value of X_n will be obtained by equating to zero the determinant

$$\begin{vmatrix} C_{p,n-1}, & C_{p,n-2}, \ldots\ldots C_{p,2}, & C_{p,1}, & X_n+Q_n \\ C_{p-1,n-2}, & C_{p\ 1,n-3}, \ldots\ldots C_{p-1,1}, & 1, & Q_{n-1} \\ C_{p-2,n-3}, & C_{p-2,n-4}, \ldots\ldots 1, & 0, & Q_{n-2} \\ C_{p-3,n-4}, & C_{p-3,n-5}, \ldots 1, & 0, & Q_{n-3} \\ \vdots & & & \\ C_{p-n+2,1}, & 1, \quad 0, \ldots\ldots 0, & 0, & Q_2 \\ 1, & 0, \quad 0, \ldots\ldots 0, & 0, & Q_1 \end{vmatrix}$$

and we learn that this determinant is equal to $X_n - 2^{2n}C_{p,2n}$. If we replace X_n by $2^{2n}C$, the determinant will vanish identically; if $2n=p$, it reduces to $X_n - 2^p$.

If now we suppose in the value $K_{2p,n}$ (p. 8), n to be an odd number, replacing it by $2n+1$, we find

$$(2) \ldots K_{2p,2n+1} = C_{2p,2n+1} - \{C_{p,1} \cdot K_{2p-2,2n-1} + C_{p,2} \cdot K_{2p-4,2n-3} + \ldots \ldots C_{p,n-1} \cdot K_{2p-2n+2,3} + C_{p,n} \cdot K_{2p-2n,1}\};$$

making $X_{n+1} = K_{2p,2n+1}$, the determinant will be

$$\begin{vmatrix} C_{p,n}, & C_{p,n-1}, \ldots\ldots C_{p,1}, & X_{n+1} - C_{2p,2n+1} \\ C_{p-1,n-1}, & C_{p-1,n-2}, \ldots\ldots 1, & -C_{2p-2,2n-1} \\ \vdots & & \\ C_{p-n+1,1}, & 1, \quad 0, \ldots\ldots 0, & -C_{2p-2n+2,3} \\ 1, & 0, \quad 0, \ldots\ldots 0, & -C_{2p-2n,1} \end{vmatrix}$$

and its value $X_{n+1} - 2^{2p+1} C_{p,2p+1}$.

We may obtain determinants involving ${}^qK_{2p+q,2n}$ and ${}^qK_{2p+q,2n+1}$ in precisely the same manner. The foregoing method applies identically to the general case, and it will be only necessary to write $2p+q$ for $2p$ in the above equations and determinants. Thus, placing

$$X'_n = {}^qK_{2p+2q,2n}, \quad Q'_n = C_{p,n} - C_{2p+q,2n},$$

we find

$$\begin{vmatrix} C_{p,n-1}, & C_{p,n-2}, \ldots\ldots C_{p,2}, & C_{p,1}, & X'_n + Q'_n \\ C_{p-1,n-2}, & C_{p-1,n-3}, \ldots\ldots C_{p-1,1}, & 1, & Q'_{n-1} \\ C_{p-2,n-3}, & \ldots\ldots 1, & 0, & Q'_{n-2} \\ \vdots & & & \\ C_{p-n+2,2}, & 1, \quad 0, \ldots\ldots 0, & 0, & Q'_2 \\ 1, & 0, \quad 0, \ldots\ldots 0, & 0, & Q'_1 \end{vmatrix}$$

and we should in the same manner obtain the determinant involving ${}^qK_{2p+q,2n+1}$.

In order to illustrate the preceding theory, we here subjoin numerical examples.

Let $n = 2$. If we use the first determinant, we shall have

$$\begin{vmatrix} C_{p,1}, & X_2 + Q_2 \\ 1, & Q_1 \end{vmatrix} \quad \text{and} \quad \begin{matrix} Q_2 = C_{p,2} - C_{2p,4} \\ Q_1 = C_{p-1,1} - C_{2p-2,2} \end{matrix}$$

and the determinant becomes

$$\begin{vmatrix} p, & X_2+\dfrac{p\,(p-1)}{2}-\dfrac{2p\,(2p-1)\,(2p-2)\,(2p-3)}{1.2.3.4} \\ 1, & p-1-\dfrac{(2p-2)\,(2p-3)}{2} \end{vmatrix}$$

But $$X_2 = K_{2p,4} = \frac{2p\,(2p-2)\,(2p-4)\,(2p-6)}{1.2.3.4};$$

hence the following determinant must vanish identically :—

$$\begin{vmatrix} p, & \dfrac{2p(2p-2)(2p-4)(2p-6)}{1.2.3.4}+\dfrac{p(p-1)}{2}-\dfrac{2p(2p-1)(2p-2)(2p-3)}{1.2.3.4} \\ 1, & p-1-\dfrac{(2p-2)\,(2p-3)}{2} \end{vmatrix}$$

Reducing and suppressing common factors,

$$\begin{vmatrix} 1, & \dfrac{(2p-4)\,(2p-6)-(2p-1)\,(2p-3)+3}{3.4} \\ 1, & -(p-2) \end{vmatrix}$$

or,

$$\begin{vmatrix} 1, & -(p-2) \\ 1, & -(p-2) \end{vmatrix}$$

If in the determinant

$$\begin{vmatrix} C_{p,1}, & X_2+Q_2 \\ 1, & Q_1 \end{vmatrix}$$

we make $p=4$, we shall have $X_2=K_{8,4}$, and we know that in this case $X_2=2^4$. Consequently the following determinant must vanish identically :—

$$\begin{vmatrix} 4, & 2^4+\dfrac{4.3}{2}-\dfrac{8.7.6.5}{2.3.4} \\ 1, & 3-\dfrac{6.5}{2} \end{vmatrix}$$

or

$$\begin{vmatrix} 1, & 2^2+\dfrac{3}{2}-\dfrac{7.5}{2} \\ 1, & 3-\dfrac{6.5}{2} \end{vmatrix} = \begin{vmatrix} 1, & -12 \\ 1, & -12 \end{vmatrix}$$

III. The well known equation $C_{m,n} = C_{m-1,n}+C_{m-1,n-1}$ can be translated into the language of restricted combinations, and will give us new expressions involving ${}^qK_{2p+q,n}$.

In the first place, if we arrange ${}^qK_{2p+q,n}$ in two groups with reference to any *free* letter, the first of which will *not* contain that letter, the second which *will contain* it, we shall find as above

$${}^qK_{2p+q,n} = {}^{q-1}K_{2p+q-1,n}+{}^{q-1}K_{2p+q-1,n-1}.$$

But if we form two similar groups with reference to any associative letter, we shall find

$$^{q}K_{2p+q,n} = {}^{q+1}K_{2p+q-1,n} + {}^{q}K_{2p+q-2,n-1};$$

hence $$^{q-1}K_{2p+q-1,n} + {}^{q-1}K_{2p-q-1,n-1} = {}^{q+1}K_{2p+q-1,n} + {}^{q}K_{2p-q-2,n-1}.$$

We might also inquire what expression corresponds to the equation

$$C_{m,n} = C_{m,m-n},$$

and this will lead us to a new class of combinations; for if we divide the product $abc\ldots a'b'c'\ldots fgh\ldots$ successively by all the combinations composing $^{q}K_{2p+q,n}$ we shall obtain combinations, *all* of which will contain *at least one* letter out of *every* pair of corresponding letters. Thus a new field for inquiry is afforded; and although no important consequences have been deduced from the above researches in this paper, yet as they have led us to a determinant, whose value it does not appear easy at first sight to obtain by actual expansion, I think the whole theory not unworthy of attention, both in itself and in its bearings upon the different branches of mathematics with which the theory of arrangements is connected.

Note.—Since this paper was read, Mr. M. Jenkins has pointed out to me the following generalisation of the problem:—

The actual combinations of a series of letters a, a', $a''\ldots$ b, $b'\ldots$ c, $c'\ldots$ h, k, l, taken n and n together, subject to the restriction that no two of the same group (e. g., a, a', a'', &c., or b, b',...&c.) shall enter into the same combination, is the coefficient of x^n in

$$\{1+(a+a'+a''+\ldots)x\}\{1+(b+b'+\ldots)x\}\{\quad\}(1+hx)(1+kx)(1+lx).$$

Their number is obtained by putting

$$a = a' = a'' = \ldots = b = b' = \ldots h \ldots = 1,$$

and is therefore the coefficient of x^n in $(1+\mu_1 x)^p(1+\mu_2 x)^q(1+\mu_3 x)^r\ldots$ if there be p groups of μ_1 letters, q of μ_2, &c. (μ_1, μ_2, ... being all unequal).

The result obtained in p. 70, Art. I., or $2^n C_{p,n}$, is therefore the coefficient of x^n in $(1+2x)^p$; and the result in p. 71, Art. II., is the coefficient of x^n in $(1+2x)^p(1+x)^q$.

The President next explained one of his Anallagmatic Squares, formed in connexion with his Theory of *Sequences*.

Mr. Woolhouse gave an abstract of the following paper:—

On General Numerical Solution. *By* W. S. B. Woolhouse, F.R.A.S.

The ultimate object of the following paper is to elicit a practical and efficient method of resolving, numerically, problems of every description involving an unknown quantity. In order to accomplish this end with the utmost generality, the most important step is to determine the inversion of a given function in a suitable and convenient series, involving the differential coefficients which appertain to an approximate value of the variable.

An approximate solution being taken as the basis or origin, let the proposed function, when developed by Taylor's Theorem, be

$$U = U_0 + U_0' . x + U_0'' . \frac{x^2}{2} + U_0''' . \frac{x^3}{2 . 3} + \ldots\ldots \quad (A),$$

where U is required to have a given value, and U_0 and the differential coefficients U_0', U_0'', U_0''', ... are known from the form of U.

Then, if $z = \dfrac{U - U_0}{U_0'}$; $c_2 = \dfrac{U''}{\lfloor 2\, U_0'}$, $c_3 = \dfrac{U'''}{\lfloor 3\, U_0'}$, ...

(A) will become, simply,

$$z = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots\ldots \quad (B);$$

and, according to the well-known formula for the reversion of an algebraic series, the evolved value of x will be expressed as follows:*

$$\begin{aligned} x = z &- c_2 . z^2 + (2c_2^2 - c_3) . z^3 + (-5c_2^3 + 5c_2 c_3 - c_4) . z^4 \\ &+ (14c_2^4 - 21c_2^2 c_3 + 6c_2 c_4 + 3c_3^2 - c_5) . z^5 \\ &+ \{-42c_2^5 + 84c_2^3 c_3 - 28(c_2^2 c_4 + c_2 c_3^2) + 7(c_2 c_5 + c_3 c_4) - c_6\} . z^6 \\ &+ \{132c_2^6 - 330c_2^4 c_3 + 60(3c_2^2 c_3^2 + 2c_2^3 c_4) - 12(c_3^3 + 3c_2^2 c_5 + 6c_2 c_3 c_4) \\ &\qquad + 8(c_2 c_6 + c_3 c_5) + 4c_4^2 - c_7\} . z^7 \\ &+ \{-429c_2^7 + 1287c_2^5 c_3 - 495(2c_2^3 c_3^2 + c_2^4 c_4) \\ &\quad + 165(c_2 c_3^3 + 3c_2^2 c_3 c_4 + c_2^3 c_5) - 45(c_2 c_4^2 + 2c_2 c_3 c_5 + c_2^2 c_6 + c_3^2 c_4) \\ &\qquad + 9(c_2 c_7 + c_3 c_6 + c_4 c_5) - c_8\} . z^8 \\ &\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \quad \ldots\ldots\ldots\ldots (C). \end{aligned}$$

The reversion of the series in this form is, however, inconveniently diffuse. Its terms consist, not merely of an infinite series, but of an unlimited number of infinite series, determined by the varied algebraic combinations of the original coefficients contained in (B). By partitioning off the different series in such manner that the successive terms of each set shall differ only in the numeral factor and in the power of the principal coefficient, we shall find that the sums of these sets of infinite series of terms can be severally exhibited in finite

* The formula is correctly given, as far as the 11th power inclusive, under the article "Reversion," in the "Penny Cyclopædia."

algebraic functions of the minor root of the quadratic equation formed by taking only the three leading terms of (A), or two of (B). And by substituting these finite functions in place of the respective sets of series which they represent, we shall thereby obtain not only an abbreviated form of reversion, but one that is necessarily more rapidly convergent, because the terms of the latter are severally condensations of a corresponding infinite series of terms contained in the former, and the more significant terms have the precedence in both.

To proceed with the subject before us, the constitution of the various isolated terms of the series (C) may be thus stated. The general term is

$$z^n \cdot \left\{ \frac{\lfloor pq+p'q'+p''q''\ldots}{\lfloor\underline{n}\,\lfloor\underline{q}\,\lfloor\underline{q'}\,\lfloor\underline{q''}\ldots} \cdot c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots\ldots \right\} \quad \ldots\ldots\ldots\ldots\ldots(\alpha),$$

where $n=1+(p-1)\,q+(p'-1)\,q'+\ldots$, and the algebraic sign is $(-1)^{q+q'+q''\ldots}$.

Consider the terms which involve powers of the principal coefficient c_2; these are hence of the form

$$\pm z^n \cdot \left\{ \frac{\lfloor 2m+pq+p'q'+p''q''\ldots}{\lfloor\underline{n}\,\lfloor\underline{m}\,\lfloor\underline{q}\,\lfloor\underline{q'}\,\lfloor\underline{q''}\ldots}\, c_2^m \cdot c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots\ldots \right\} \quad \ldots\ldots\ldots(\beta),$$

and the terms which do not contain c_2 are likewise included by beginning with $m=0$.

Put $\quad r=pq+p'q'+p''q''\ldots\ldots, \quad s=q+q'+q''\ldots\ldots$

Then $n=m+(r-s+1)$; and the general term is

$$\pm \frac{\lfloor 2m+r}{\lfloor\underline{m+r-s+1}\,\lfloor\underline{m}\,\lfloor\underline{q}\,\lfloor\underline{q'}\,\lfloor\underline{q''}\ldots}\, (c_2 z)^m\, z^{r-s+1}\, c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots\ldots ;$$

and putting $v=c_2 z$, and collecting these terms for all values of m, we get

$$\frac{c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''}\ldots}{\lfloor\underline{q}\,\lfloor\underline{q'}\,\lfloor\underline{q''}\ldots}\, z^{r-s+1} \Sigma \frac{\lfloor 2m+r}{\lfloor\underline{m+r-s+1}\,\lfloor\underline{m}}\, (-v)^m \quad \ldots\ldots\ldots\ldots (\gamma).$$

As a preliminary to the finite evaluation of this infinite series, first take the initial case in which the terms exhibit powers of c_2 only; then $q=0$, $q'=0$, &c., and the complete sum of this set of terms is

$$z \cdot \Sigma \frac{\lfloor 2m}{\lfloor\underline{m+1}\,\lfloor\underline{m}}\, (-v)^m = z\, \frac{\sqrt{1+4v}-1}{2v},$$

since in this last expression the coefficient of v^m is obviously half the coefficient of v^{m+1} in $\sqrt{1+4v}$.

The expression just found is a root of the quadratic $z=x+c_2 x^2$, as it evidently ought to be.

Let $\quad \alpha=\dfrac{\sqrt{1+4v}-1}{2v}, \quad \beta=\dfrac{1}{\sqrt{1+4v}}, \quad \gamma=\dfrac{1-\beta}{2}, \quad \omega=\alpha z,$

$$h_3=\beta c_3, \quad h_4=\beta c_4, \quad h_5=\beta c_5, \quad \&c.$$

Then, since $\alpha^2 = \frac{1-a}{v}$, if ϕ be an arbitrary function,

$$\alpha^{r+2}\phi = \frac{1}{v}(\alpha^r\phi - \alpha^{r+1}\phi).$$

Therefore, if $f_{r,m}$ denote the coefficient of v^m in $\alpha^r\phi$, and r take consecutive integer values, the coefficients will be thus related:

$$f_{r+2,m} = -\Delta f_{r,m+1}.$$

This is satisfied by either of the forms of coefficients $r\frac{\lfloor 2m+r-1}{\lfloor m+r \rfloor m}$ and $\frac{\lfloor 2m+r}{\lfloor m+r \rfloor m}$. To deduce the corresponding values of ϕ, make $r=1$, and the former gives $a\phi = a$ or $\phi = 1$; also, when $r=0$, the latter gives $\phi = \beta$.

Hence in α^r the coefficient of v^m is $r\frac{\lfloor 2m+r-1}{\lfloor m+r \rfloor m}$,

„ $\alpha^r\beta$ „ „ $\frac{\lfloor 2m+r}{\lfloor m+r \rfloor m}$;

therefore $\alpha^r = \Sigma r\frac{\lfloor 2m+r-1}{\lfloor m+r \rfloor m}(-v)^m$, $\alpha^r\beta = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r \rfloor m}(-v)^m$.

We shall now determine the functions exhibited in (γ), viz.,

$$\psi_{r,s} = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r-s+1 \rfloor m}(-v)^m.$$

In the first two cases, $s=0$, $s=1$, we have

$$\psi_{r,0} = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r+1 \rfloor m}(-v)^m = \frac{\alpha^{r+1}}{r+1},$$

$$\psi_{r,1} = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r \rfloor m}(-v)^m = \alpha^r\beta.$$

Put $$\mathrm{R} = \frac{(av)^{r+1}}{r+1} = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r+1 \rfloor m}(-v)^{m+r+1};$$

then $$\frac{d^s\mathrm{R}}{dv^s} = \Sigma\frac{\lfloor 2m+r}{\lfloor m+r-s+1 \rfloor m}(-v)^{m+r-s+1}$$

$$= v^{r-s+1}\psi_{r,s}.$$

Now, taking the expression $\mathrm{R} = \frac{(av)^{r+1}}{r+1}$ and differentiating successively, observing that $\frac{d(av)}{dv} = \beta$, $\frac{d\beta}{dv} = -2\beta^3$; and also $2\beta av = 1-\beta$, we get

$$\frac{d\mathrm{R}}{dv} = (av)^r \beta,$$

$$\frac{d^2\mathrm{R}}{dv^2} = r\,(av)^{r-1}\beta^2 - 2\,(av)^r\beta^3$$
$$= (av)^{r-1}\beta^2\,(r - 2\beta av) = (av)^{r-1}\beta^2\,(r - 1 + \beta),$$

$$\frac{d^3\mathrm{R}}{dv^3} = (r-1)\,(av)^{r-2}\beta^3\,(r-1+\beta) - 2\,(av)^{r-1}\beta^3\,\{2\,(r-1)\,\beta + 3\beta^2\}$$
$$= (av)^{r-2}\beta^3\,\{(r-1)^2 + (r-1)\,\beta - (1-\beta)\,[2\,(r-1) + 3\beta]\}$$
$$= (av)^{r-2}\beta^3\,\{(r-1)\,(r-3) + 3\,(r-2)\,\beta + 3\beta^2\},$$

$$\frac{d^4\mathrm{R}}{dv^4} = (r-2)\,(av)^{r-3}\beta^4\,\{(r-1)\,(r-3) + 3\,(r-2)\,\beta + 3\beta^2\}$$
$$- 2\beta^3\,(av)^{r-2}\,\{3\,(r-1)\,(r-3)\,\beta^2 + 12\,(r-2)\,\beta^3 + 15\beta^4\}$$
$$= (av)^{r-3}\beta^4\,[(r-1)\,(r-2)\,(r-3) + 3\,(r-2)^2\,\beta + 3\,(r-2)\,\beta^2$$
$$- (1-\beta)\,\{3\,(r-1)\,(r-3) + 12\,(r-2)\,\beta + 15\beta^2\}]$$
$$= (av)^{r-4}\beta^4\,\{(r-1)\,(r-3)\,(r-5) + (6r^2 - 36r + 45)\,\beta$$
$$+ 15\,(r-3)\,\beta^2 + 15\beta^3\}\ ;\ \&c.\ \&c.$$

After putting $1 - 2\gamma$ for β in the last factor of each expression, we obtain

$$\psi_{r,1} = \alpha^r\beta,$$
$$\psi_{r,2} = \alpha^{r-1}\beta^2\,\{r - 2\gamma\},$$
$$\psi_{r,3} = \alpha^{r-2}\beta^3\,\{r\,(r-1) - 6r\gamma + 12\gamma^2\},$$
$$\psi_{r,4} = \alpha^{r-3}\beta^4\,\{r\,(r-1)\,(r-2) - 12r\,(r-1)\,\gamma + 60r\gamma^2 - 120\gamma^3\},$$
$$\&c. \qquad \&c. \qquad \&c.$$

And, generally,

$$\psi_{r,s} = \alpha^{r-s+1}\beta^s \left\{ (r \ldots r-s+2) - \frac{s\,(s-1)}{1}\,(r \ldots r-s+3)\,\gamma \right.$$
$$\left. + \frac{s+1 \ldots s-2}{1.2}\,(r \ldots r-s+4)\,\gamma^2 - \frac{s+2 \ldots s-3}{1.2.3}\,(r \ldots r-s+5)\,\gamma^3 \ldots \right\} \ldots (\delta).$$

Hence, observing that $\alpha^{r-s+1}\,z^{r-s+1} = \omega^{r-s+1}$,

$$\beta^s \cdot c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots = \beta^{q+q'+q''\ldots}\, c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots = h_p^q\, h_{p'}^{q'}\, h_{p''}^{q''} \ldots,$$

the sum of the terms (β), viz.

$$\Sigma \pm z^n \cdot \left\{ \frac{\lfloor 2m + pq + p'q' + p''q'' \ldots}{\lfloor n \lfloor m \lfloor q \lfloor q' \lfloor q'' \ldots}\, c_2^m \cdot c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots \right\},$$

when m takes all values 0, 1, 2, ... ∞, is

$$z^{r-s+1}\,\frac{c_p^q\, c_{p'}^{q'}\, c_{p''}^{q''} \ldots}{\lfloor q \lfloor q' \lfloor q'' \ldots}\,\psi_{r,s} = \omega^{r-s+1} \cdot h_p^q\, h_{p'}^{q'}\, h_{p''}^{q''} \ldots \frac{\lfloor s-1}{\lfloor q \lfloor q' \lfloor q'' \ldots}$$
$$\times \left\{ \frac{r \ldots r-s+2}{1 \ldots s-1} - \frac{s}{1}\,\frac{r \ldots r-s+3}{1 \ldots s-2}\,\gamma + \frac{s\,(s+1)}{1.2}\,\frac{r \ldots r-s+4}{1 \ldots s-3}\,\gamma^2 \right.$$
$$\left. - \frac{s\,(s+1)\,(s+2)}{1.2.3}\,\frac{r \ldots r-s+5}{1 \ldots s-4}\,\gamma^3 + \&c. \right\} \ldots\ldots\ldots (\epsilon),$$

which is finite and consists of s terms. The numeral coefficients within the brackets are simply formed by writing down s terms of the series $1,\ s,\ \frac{s(s+1)}{2},\ \frac{s(s+1)(s+2)}{2.3}$, &c., and placing underneath them s terms of the series $1,\ r,\ \frac{r(r-1)}{2},\ \frac{r(r-1)(r-2)}{2.3}$, &c., but in a reverse order, and then multiplying together the corresponding numbers vertically.

It will now be easy to write down *ad libitum* the collective values for any proposed sets of terms of the development (C). We shall here enumerate some of the principal cases, and then state the complete formula.

The leading set of terms are those which involve c_2 only, viz.,

$$z-z^2c_2+2z^3c_2^2-5z^4c_2^3+\ldots,$$

which are of the form $\lambda z^n c_2^m$. In this case $r=0$, $s=0$, and the sum of all the terms referred to has before been found $=az=\omega$.

The next set $\quad -z^3.\,c_3+5z^4c_2\,.\,c_3-21z^5c_2^2.\,c_3+\ldots$

comes under the form $\lambda z^n c_2^m\,.\,c_3$. Here $p=3$, $q=1$; therefore $r=3$, $s=1$, and the sum of the terms $= z^3\,\frac{c_3}{[1}\,\psi_{3,1} = z^3c_3\,.\,\alpha^3\beta = \omega^3 h_3$.

Take now the set

$$7z^6c_3c_4-72z^7c_2c_3c_4+495z^8c_2^2c_3c_4-\ldots$$

These are of the form $\lambda z^n c_2^m\,.\,c_3c_4$, in which $r=7$, $s=2$; and therefore the sum $= z^6c_3c_4\,.\,\psi_{7,2} = z^6c_3c_4\,.\,\alpha^6\beta^2(7-2\gamma) = \omega^6.\,h_3h_4(7-2\gamma)$.

As one more example, take the terms which are of the form $\lambda z^m c_2^m\,.\,c_3^2c_4$. In this case, $c_p^q\,c_{p'}^{q'} = c_3^2\,c_4^1$; therefore $r=10$, $s=3$, and the sum of all such terms $= \omega^8.\,h_3^2h_4(45-30\gamma+6\gamma^2)$.

By taking in this way every requisite form of combination of the coefficients, and afterwards uniting the several values, the development (C) becomes concentrated into the following comprehensive result:

$$\begin{aligned}
x = \quad & \omega \\
& -\omega^3.\,h_3 \\
& -\omega^4.\,h_4 \\
& -\omega^5\{h_5-h_3^2(3-\gamma)\} \\
& -\omega^6\{h_6-h_3h_4(7-2\gamma)\} \\
& -\omega^7\{h_7-(2h_3h_5+h_4^2)(4-\gamma)+h_3^3(12-9\gamma+2\gamma^2)\} \\
& -\omega^8\{h_8-(h_3h_6+h_4h_5)(9-2\gamma)+h_3^2h_4(45-30\gamma+6\gamma^2)\} \\
& - \quad \&c. \qquad \&c. \qquad \&c. \quad \ldots\ldots\ldots\

the 8th order of ω, and not only includes the values of all the terms exhibited in the development (C), but embodies all the cognate terms to infinity. Now the value of γ depends upon the coefficient h_2, and is

$$\gamma = \beta\alpha v = \beta c_2 \omega = h_2 \omega.$$

Therefore, by substitution and retention of all terms to the 8th order, we obtain

$$\begin{aligned} x = \omega &- \omega^3 . h_3 \\ &- \omega^4 . h_4 \\ &- \omega^5 . (h_5 - 3h_3^2) \\ &- \omega^6 . (h_6 - 7h_3h_4 + h_2h_3^2) \\ &- \omega^7 . (h_7 - 8h_3h_5 - 4h_4^2 + 2h_2h_3h_4 + 12h_3^3) \\ &- \omega^8 . (h_8 - 9h_3h_6 - 9h_4h_5 + 45h_3^2h_4 + 2h_2h_3h_5 + h_2h_4^2 - 9h_2h_3^2) \\ &\quad \&c. \qquad \&c. \qquad \&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (E). \end{aligned}$$

By means of (ϵ) we find that the general term of (E) is

$$(-1)^{s-m}\omega^n . \frac{\lfloor s+m-1}{\lfloor m \, \lfloor s-m-1}\, \frac{\lfloor r}{\lfloor n \, \lfloor q \, \lfloor q' \ldots}\, h_2^m h_p^q h_{p'}^{q'} \ldots$$

where $r = pq + p'q' + \ldots$, $s = q + q' + \ldots$, $n = m + (r - s + 1)$;

and where h_2 is absent, or $m = 0$, the terms are invariably the same as the analogous terms contained in the ordinary formula (C).

The formula, however, yet admits of further concentration and simplification. Referring back to the original equation

$$z = x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots,$$

we perceive that ω is a solution of the quadratic portion

$$z = \omega + c_2\omega^2,$$

and that this would be an accurate solution of the entire equation if we could assume that the residue $c_3\omega^3 + c_4\omega^4 + \ldots = 0$, or $h_3\omega^3 + h_4\omega^4 + \ldots = 0$. Thus the expression for $\omega - x$, or the collected terms of the development which involve exclusively the coefficients h_3, h_4, &c., must vanish with $h_3\omega^3 + h_4\omega^4 + \ldots$, and therefore contain the latter as a factor. Hence, putting

$$\mu = h_3\omega^3 + h_4\omega^4 + h_5\omega^5 + \ldots,$$

the foregoing expression is readily transformed into

$$\begin{aligned} x = \omega &- \mu - h_2\mu^2 \\ &+ 3h_3\mu\omega^2 \\ &+ 4h_4\mu\omega^3 \\ &+ (5h_5 - 12h_3^2)\,\mu\omega^4 \\ &+ (6h_6 - 33h_3h_4 + 9h_2h_3^2)\,\mu\omega^5 \ldots\ldots\ldots (F). \end{aligned}$$

In practice, however, we may simply take $x = \omega - \mu$; the other terms are of the fifth and higher orders, and may be omitted as immaterial.

It may be useful here to recapitulate the principal conditions and formulæ for practical application. The symbol U denoting the function to be reverted, the equation to be solved with respect to x is $U=0$. Then U', U'', U''', &c. being the successive differential coefficients, the formulæ for calculation will be

$$\left.\begin{aligned} z &= -\frac{U_0}{U'};\ c_2 = \frac{1}{\lfloor 2}\frac{U''}{U'},\quad c_3 = \frac{1}{\lfloor 3}\frac{U'''}{U'},\ \&c. \\ v &= c_2 z, \qquad q = \sqrt{1+4v}, \\ p &= \frac{1+q}{2}, \qquad \omega = \frac{z}{p} \\ h_3 &= \frac{c_3}{q}, \qquad h_4 = \frac{c_4}{q},\ \&c. \\ x &= \omega - h_3\omega^3 - h_4\omega^4 \ldots \end{aligned}\right\} \ldots\ldots(\lambda).$$

As the development proceeds in powers of ω or z, the efficacy of the approximation requires that z may be a small fraction, or that U_0 may be small as compared with U'_0. That is, the zero or starting-point of the calculation should be taken as near as practicable to the sought value.

Should the function be of such a nature that its differential coefficients are of a complicated character, it will be more convenient to compute a series of values of the function for assumed equidistant values of x, and thence to determine the differential coefficients as depending upon the several orders of differences of these quantities. In this method the convergency of the differences will indicate if the interval be taken sufficiently small; and that quantity which comes nearest to the prescribed result being taken as the origin or zero-point of the calculation, the accuracy of the process will be satisfactorily established. The computation of five values of the function, so as to include two preceding and two following the origin, will in general supply all the necessary data. If these values and their differences be

$$\begin{array}{l|l|l|l|l} U_{-2} & & & & \\ U_{-1} & a_{-2} & & & \\ U_0 & a_{-1} & b_{-1} & & \\ U_{+1} & a_{+1} & b_0 & c_{-1} & \\ U_{+2} & a_{+2} & b_{+1} & c_{+1} & d_0, \end{array}$$

and if, moreover,

$$a_0 = \tfrac{1}{2}(a_{-1}+a_{+1}),$$
$$c_0 = \tfrac{1}{2}(c_{-1}+c_{+1}),$$

the differential coefficients of the function at the central value U_0 will be

$$U' = a_0 - \frac{1}{2.3} c_0,$$

$$U'' = b_0 - \frac{1}{3.4} d_0,$$

$$U''' = c_0, \qquad U'''' = d_0.$$

These being used as data according to the formula (λ), the resulting value of x will be expressed in parts of the interval originally chosen for the five computed values of the function.

Example 1.—The differences between the longitudes of the sun and moon on certain consecutive days in May, 1869, are as hereunder stated:

1869	☾ − ☉	Differences			
May 9	−25° 1′ 23″·4	+11° 23′ 34″·4	+13′ 13″·8	+ 1″·8	−38″·3
10	−13 37 49·0	11 36 48·2	13 15·6	−36·5	
11	− 2 1 0·8	11 50 3·8	+12 39·1		
12	+ 9 49 3·0	+12 2 42·9			
13	+21 51 45·9				

From these find the time of new moon or the exact time when ☾ − ☉ is zero.

Here we have

$a_0 = +11° 43′ 26″·0$, $b_0 = +13′ 15″·6$, $c_0 = -17″·3$, $d_0 = -38″·3$;
and $-U_0 = +2° 1′ 0″·8$, $U' = +11° 43′ 28″·9$, $U'' = +13′ 18″·8$,

which being converted into seconds, the calculation is as follows:

log U″	+ 2·9024381	log (−U₀)	+3·8609845
,, U′	+ 4·6254041	.....	+4·6254041
,, 2c₂	+ 8·2770340	log z	+9·2355804
,, 2	0·3010300	..... ,, 4c₂	+8·5780640
Nat. Nos.	·0065109	,, 4v	+7·8136444
	1·0065109	,, 1+4v	0·0028185
	1·0032501	,, q	0·0014092
	1·0016250	,, p	0·0007052 *
	0·1717415	,, ω	9·2348752

The time of new moon, or conjunction in longitude, is therefore May 11^d·171745, that is, on May 11th at 4^h 7^m 19^s P.M.

The terms $-h_3\omega^3 - h_4\omega^4$ would not sensibly affect the result, which they would increase by only three-hundredth parts of a second.

* Here, if $\log p = \frac{1}{2}\log q + \text{corr}^n$, the small correction might be conveniently tabulated to arg. $\frac{1}{2}\log q$.

Example 2.*—The Long Annuities, which have 30 years to run, are sold at 19 years' purchase; what rate of interest does the purchaser obtain for his money?

In this example the equation for solution is

$$U = \frac{1-(1+\rho)^{-30}}{\rho} - 19 = 0,$$

which does not admit of direct algebraic solution by any known method. But if we take any specified rate of interest, it is easy to calculate the corresponding years' purchase from the formula $\frac{1-(1+\rho)^{-30}}{\rho}$. In fact, these results are already calculated and registered in extensive tables to be found in all works on Compound Interest and Annuities. From these the following values, for each half per cent. of interest, are extracted and differenced:

ρ	Years' purchase	Differences			
2	22·396456				
		−1·466163			
$2\frac{1}{2}$	20·930293		+·136311		
		1·329852		−14855	
3	19·600441		·121456		+1783
		1·208396		−13072	
$3\frac{1}{2}$	18·392045		+·108384		
		−1·100012			
4	17·292033				

We have to calculate from these the rate of interest at which the year's purchase becomes exactly 19; and for this calculation they supply the following data:

$$-U_0 = -·600441, \quad U' = -1·266797, \quad U'' = +·121307,$$
$$U''' = -·013963, \quad U'''' = +·001783.$$

Hence, as before, by the formula (λ) we find

$$x = \omega - h_3\omega^3 - h_4\omega^4 = ·4852579$$
$$-2201$$
$$+34$$
$$= ·4850412$$

This being expressed in parts of the interval of half per cent., we must take one-half of it, so that the required rate of interest is 3·2425206 per cent. per annum, which is true to the last place of figures.

Example 3.—As a concluding example, take a cubic equation, given in most treatises on Algebra, viz., $x^3 - 2x - 5 = 0$.

* From David Jones's Annuities, &c., p. 42.

By successive differentiation, we have

$$\begin{aligned} U &= x^3-2x-5, \\ U' &= 3x^2-2, \\ U'' &= 6x, \\ U''' &= 6. \end{aligned}$$

A real root of the equation evidently lies near to the number 2; and for $x=2$ we have

$$-U=1, \quad U'=10, \quad U''=12, \quad U'''=6;$$

$$\therefore \quad z=\frac{1}{10}, \quad c_2=\frac{3}{5}, \quad c_3=\frac{1}{10};$$

$$v=\frac{3}{50}, \quad q=\frac{\sqrt{31}}{5}, \quad p=\frac{\sqrt{31}+5}{10},$$

$$\omega=\frac{\sqrt{31}-5}{6}, \qquad h_3=\frac{1}{2\sqrt{31}};$$

$$\begin{aligned} \therefore \quad x &= 2+\omega-h_3\omega^3 \\ &= 2+\frac{\sqrt{31}-5}{6}-\frac{1}{2\sqrt{31}}\left(\frac{\sqrt{31}-5}{6}\right)^3 \\ &= \frac{1}{216}\left(\frac{1411}{31}\sqrt{31}+199\right) \\ &= 2{\cdot}0945513, \text{ which is true to the last figure.} \end{aligned}$$

These examples will serve to show the practical utility of the formula, which may with unfailing efficacy be similarly applied to all kinds of problems. It is, however, in the numerical solution of complicated and otherwise unmanageable equations that the value of the method will be most conspicuous. All complication is entirely got rid of so soon as the successive numerical values are obtained by direct computation.

Remark on the preceding Paper. *By* A. De Morgan, Esq., F.R.A.S.

Having seen Mr. Woolhouse's Paper before it was publicly read, I arrived at an independent establishment of the development which is, in the Paper, deduced from the known series for reversion. This mode of establishment is also an extension; though it is hardly to be expected that any extended application will be found desirable. At the request of Mr. Woolhouse, I subjoin a short notice.

Let $\phi\omega=\varphi x+\psi x$, where ψx will be small; and write fx, when convenient, for $\varphi x+\psi x$. Assume $x=\omega+t$, which gives

$$-\psi\omega=f'\omega\,.\,t+f''\omega\,.\,\frac{t^2}{2}+\ldots.$$

Find t in powers of $-\psi\omega$ by common inversion, omitting ω after f for abbreviation, and writing f_n for $f^{(n)} : 2.3...n$,

$$x = \omega - \frac{1}{f_{,}} \cdot \psi\omega - \frac{f_{,,}}{f_{,}^3} \cdot (\psi\omega)^2 - \frac{2f_{,,}^2 - f_{,} f_{,,,}}{f_{,}^5} \cdot (\psi\omega)^3 - \frac{5f_{,,}^3 - 5f_{,} f_{,,} f_{,,,} + f_{,}^2 f_{,,,,}}{f_{,}^7} \cdot (\psi\omega)^4 - \ldots\ldots,$$

a representation of $f^{-1}\phi\omega$, or of $f^{-1}(f\omega - \psi\omega)$. Three terms will be more than sufficient.

Let $\phi x + \psi x$ be $\quad ax + bx^2 + \ldots + kx^m + (p + qx + \ldots + sx^n)\, x^{n+1}$;

let $\alpha = \phi x + \psi x$ be the equation to be solved, and let $\alpha = \phi x$ give $x = \omega$. The root of $\alpha = \phi\omega$ being known, $\phi'\omega$ is known; call it l. Expanding the above in powers of ω, it is clear that the first three terms will give all short of ω^{3m+3}; Mr. Woolhouse, by an entirely different method, stops at ω^8 when $m = 2$. Taking this case, we find $\alpha = a\omega + b\omega^2$, $a + 2b\omega = \surd(a^2 + 4b\alpha) = l$. And, writing down no more than necessary for ω^8 inclusive,

$$x = \omega - \frac{c + e\omega + f\omega^2 + g\omega^3 + h\omega^4 + k\omega^5}{l + 3c\omega^2 + 4e\omega^3 + 5f\omega^4 + 6g\omega^5 + 7h\omega^6 + 8k\omega^7}\, \omega^3 - \frac{(b + 3c\omega + 6e\omega^2 + \ldots + 28k\omega^6)\,(c + e\omega + f\omega^2)^2}{(l + 3c\omega^2 + \ldots + 8k\omega^7)^3}\, \omega^6 - \ldots.$$

For l write 1; and, remembering that in this result al^{-1}, bl^{-1}, &c. must be written for a, b, &c., we have

$$x = \omega - c\omega^3 - e\omega^4 - (f - 3c^2)\,\omega^5 - (g - 7ce + bc^2)\,\omega^6 - (h - 8cf - 4e^2 + 12c^3 + 2bce)\,\omega^7 - (k - 9cg - 9ef + 45c^2e - 9bc^3 + 2bcf + be^2)\,\omega^8 - \ldots.$$

This result agrees entirely with that of Mr. Woolhouse.

The following Paper, read May 28th, 1868, could not be inserted in the account of the Proceedings of that day :—

On some Geometrical Constructions.

By H. J. Stephen Smith, F.R.S., Savilian Professor of Geometry in the University of Oxford.

Art. 1. A conic A is said to *circumscribe harmonically* a conic B, when A circumscribes a triangle which is self-conjugate with regard to B. Similarly, A is said to be *inscribed harmonically in* B, when A is

inscribed in a triangle which is self-conjugate with regard to B. Though this mode of expression is not very accurate, it has the advantage of brevity, and it may serve to fix in the memory the well known theorems— .

I. "If A circumscribe B harmonically, B is harmonically inscribed in A."

II. "If A circumscribe B harmonically, the conic corresponding to A in any correlative figure is harmonically inscribed in the conic corresponding to B."

Of these theorems we shall have to make frequent use, as also of the two following and their correlatives:—

III. "If A circumscribes B harmonically, A circumscribes an infinite number of triangles self-conjugate with regard to B; viz., if x_1 is any point of A, and x_2x_3 is the chord intercepted by A on the polar of x_1 with regard to B, $x_1x_2x_3$ is a self-conjugate triangle with regard to B. Or, which is the same thing, the *harmonic envelope of* A *and* B (*i. e.* the conic enveloped by lines cutting A and B harmonically) coincides with the polar reciprocal of A with regard to B."

IV. "If A circumscribe B harmonically, the centre of homology of any triangle inscribed in A, and of its polar triangle with regard to B, will lie on A."—(Dr. Salmon's *Conic Sections*, p. 326.)

The pairs of points, conjugate with regard to a conic A, which lie upon a line L, form a system in involution. Similarly, the pairs of lines, conjugate with respect to A, which intersect at a point P, form a pencil in involution. These involutions we shall term *the involutions of* A *upon the line* L, *and at the point* P, respectively. When we say that a conic is given, we shall understand that the polar system of the conic is given; *i. e.*, that the involution of the conic upon any line, and at any point in its plane, is given, or can be determined linearly. If any single element (point or tangent) of a given conic is given, we can determine linearly as many elements of it as we please. But if no single element of a given conic is given, the determination of any single element will require a quadratic construction, and the conic itself may be imaginary.

When there are two involutions I_1 and I_2 upon the same line L, the involution of which the double points are the extremities of the segment common to I_1 and I_2, is said to be *compounded of* I_1 *and* I_2. To obtain the involution compounded of two given involutions I_1 and I_2, let x be any point of L; x_1, x_2 the conjugates of x in I_1 and I_2 respectively, y_1 the conjugate of x_2 in I_1, y_2 the conjugate of x_1 in I_2: the harmonic conjugate of x with regard to y_1y_2 is also the conjugate of x in the involution compounded of I_1 and I_2. This construction may be demonstrated by projecting the involutions I_1 and I_2 upon a conic in

the usual manner, and applying Pascal's theorem to the pentagon formed by the projections of the points x, x_1, y_2, y_1, x_2, x.

All the constructions which we shall employ in this paper are linear, except when the contrary is expressly stated. We shall, for the most part, leave the correlative of each proposition to be supplied by the reader.

ART. 2. *Problem* 1.—"To determine the conic σ, which passes through three given points a, b, c, and circumscribes harmonically two given conics S_1 and S_2."

Solution.—Let $a_1 b_1 c_1$, $a_2 b_2 c_2$ be the polar triangles of abc with regard to S_1 and S_2 respectively; and let aa_1, bb_1, cc_1 meet in x_1; aa_2, bb_2, cc_2 in x_2; then x_1, x_2 are points of σ (Theorem IV., art. 1), which is thus completely determined by the five points a, b, c, x_1, x_2.

Problem 2.—"To determine the conic σ, which passes through two given points a, b, and circumscribes harmonically three given conics S_1, S_2, S_3."

Solution.—Let c_1, c_2, c_3 be the poles of ab with regard to S_1, S_2, S_3 respectively. Through a draw any line aP, not passing through any one of the points c_1, c_2, c_3; let p_1, p_2, p_3 be the poles of aP with regard S_1, S_2, S_3; and let P be the unknown point in which σ meets aP for the second time. Considering the triangle abP, inscribed in σ, with regard to each of the conics S_1, S_2, S_3 in succession, we see that the three intersections (Pc_1, bp_1), (Pc_2, bp_2), (Pc_3, bp_3), as well as a, b, P, lie upon σ. We have therefore the anharmonic equation

$$P.[a, c_1, c_2, c_3] = b.[a, p_1, p_2, p_3],$$

which implies that the conic passing through a, c_1, c_2, c_3, and satisfying the anharmonic equation

$$[a, c_1, c_2, c_3] = b.[a, p_1, p_2, p_3],$$

also passes through P. Thus P is determined linearly, and with it σ, on which we now have six points. (The actual construction of P is as follows:—Let aP cut c_2c_3 in a'; determine on c_2c_3 a point c', satisfying the anharmonic equation

$$[a', c', c_2, c_3] = b.[a, p_1, p_2, p_3];$$

the point P is the intersection of aP and $c'c_1$.)

Problem 3.—"To determine the conic σ, which passes through a given point a, and circumscribes harmonically four given conics S_1, S_2, S_3, S_4."

Solution.—Let aP, aQ be any two straight lines passing through a, but not conjugate with regard to any one of the given conics. Let $p_1p_2p_3p_4$, $q_1q_2q_3q_4$ be the poles of aP, aQ respectively with regard to the conics S_1, S_2, S_3, S_4; and let P, Q be the unknown points in which aP, aQ meet σ for the second time. Considering the triangle aPQ, inscribed

in σ, with regard to the conics S_1, S_2, S_3, S_4 in succession, we obtain the anharmonic equation

$$P \cdot [a, q_1, q_2, q_3, q_4] = Q \cdot [a, p_1, p_2, p_3, p_4],$$

which suffices for the linear determination of P and Q, by a kind of double position. Assume any point x on aP as the true position of P, and determine on aQ the corresponding position y of Q, first by the equation

$$y_3 \cdot [a, p_1, p_2, p_3] = x \cdot [a, q_1, q_2, q_3],$$

and then by the equation

$$y_4 \cdot [a, p_1, p_2, p_4] = x \cdot [a, q_1, q_2, q_4].$$

The two positions of y thus obtained will form, when x varies, two homographic ranges y_3 and y_4; of the double points, one is at the intersection of aQ and p_1p_2, the other can be determined linearly, and is the true position of Q. The details of the construction are as follows:—Denote by q the intersection $(a\text{P}, q_1q_2)$, and by p the intersection $(a\text{Q}, p_1p_2)$; let x be a point varying its position on aP, and let xq_3, xq_4 intersect q_1q_2 in $q'_3q'_4$. On the line p_1p_2 determine the points p'_3, p'_4, which satisfy the equation

$$[p'_3, p'_4, p, p_1, p_2] = [q'_3, q'_4, q, q_1, q_2],$$

and let $p_3p'_3$, $p_4p'_4$ cut aQ in y_3, y_4 respectively. The points y_3, y_4 will form two homographic ranges on aQ; for each of these ranges is homographic with the range x. One of the double points of the two ranges is at p; for if we imagine x to coincide with q, q'_3 and q'_4 will coincide with one another, and with q; whence p'_3 and p'_4, and with them y_3 and y_4, will coincide with one another, and with p. The other double point can therefore be obtained linearly; it will be the point Q, and the corresponding position of the point x will be the point P. We shall thus have seven points on σ; viz., the three points a, P, Q, and the four intersections $(\text{P}q_r, \text{Q}p_r)$, where $r = 1, 2, 3, 4$.

Problem 4.—"To determine the conic σ, which circumscribes harmonically five given conics S_1, S_2, S_3, S_4, S_5." Only the polar system of σ can be determined linearly, and σ itself may be imaginary.

Solution.—(1°). If each of two conics A and B harmonically circumscribe a third S, every conic C, which passes through the intersections of A and B, also circumscribes S harmonically.* To prove this geometrically, let x be any one of the points of intersection of A and B, and let the polar of x with respect to S cut A, B, C in the points a_1a_2, b_1b_2, c_1c_2 respectively. Then a_1a_2 and b_1b_2 are pairs of conjugate points with respect to S; therefore also c_1c_2, which is in involution

* The theorem of M. Hesse, "If two pair of opposite vertices of a quadrilateral are conjugate pairs with regard to a conic, the third pair of opposite vertices is also a conjugate pair," is a particular case of the correlative theorem.

with a_1a_2 and b_1b_2, is a pair of conjugate points with respect to S; *i. e.*, C circumscribes a triangle xc_1c_2 which is self-conjugate with respect to S. From this it appears that the conics which harmonically circumscribe four given conics all pass through four fixed points; for, if A, B are two conics circumscribing S_1, S_2, S_3, S_4, the conic C, which passes through a given point c, and circumscribes harmonically those four conics, is no other than the conic of the system (A, B) which passes through the point c.

(2°). Let L be any line in the plane of the five given conics: to obtain the involution of σ upon L, we have first to determine the intersections of L by two of the conics which harmonically circumscribe the four conics S_2, S_3, S_4, S_5; and to do this, we have only, in Problem 3, to take successively for a two different points on L. Let I_1 represent the involution determined by the two pairs of intersections; similarly, let I_2 represent the involution determined on L by the system of conics which circumscribe harmonically the four conics S_1, S_3, S_4, S_5. The involution of σ upon L is the involution compounded of I_1 and I_2.

Art. 3. In the preceding problems, any number of the given conics may degenerate into pairs of points, but the solutions will remain applicable if we observe that the pole of a straight line L, with regard to a system of two points, is the harmonic conjugate, with regard to the two points, of the intersection of L by the line joining the two points. Since a conic circumscribing a conic, which has degenerated into a pair of points, is a conic with regard to which the two points are conjugate, our last problem includes that of M. de Jonquières ("Annales de Mathématiques, par MM. Terquem et Gerono, vol. xiv. p. 435), "To determine the conic which divides harmonically five given segments." Again, the two points of a degenerate conic may become coincident, in which case a conic circumscribing harmonically the degenerate conic, is simply a conic passing through the point which represents that conic. Thus the problem 4 includes the problems 1, 2, 3. We might have made the solution of these problems depend on the corresponding cases of M. de Jonquières' problem. For example, if x_1y_1, x_2y_2, x_3y_3 x_4y_4 are the polar chords of the point a with regard to the conics S_1, S_2, S_3, S_4, the conic passing through a, and dividing harmonically the four segments x_1y_1, x_2y_2, x_3y_3, x_4y_4, harmonically circumscribes the conics S_1, S_2, S_3, S_4. But the solutions which M. de Jonquières has given of the various cases of his problem are, perhaps, less direct than those which we have deduced from the theorem of Dr. Salmon (Theorem 4, Art. 1.)

We may add that it follows from the solution of the problem of M. de Jonquières, that the polar system of a conic is given, when five pair of conjugate points with regard to the conic are given (but these five pair must be independent; see Art. 4).

Art. 4. Some remarks, which may not be without interest, are suggested by the analysis corresponding to the problems 1—4.

Let $$\alpha_i x^2+\beta_i y^2+\gamma_i z^2+2\alpha'_i yz+2\beta'_i xz+2\gamma'_i xy = 0 \ldots\ldots\ldots(1)$$
be the equation in point coordinates of the conic σ_i, and let
$$A_i\xi^2+B_i\eta^2+C_i\zeta^2+2A_i\eta\zeta+2B'_i\xi\zeta+2C'_i\xi\eta = 0 \ldots\ldots\ldots(2)$$
be the equation in line coordinates of the conic S_i, the two sets of coordinates being connected by the relation
$$\xi x+\eta y+\zeta z = 0.$$
The equation which expresses that σ harmonically circumscribes S_i is
$$A_i\alpha+B_i\beta+C_i\gamma+2A'_i\alpha'+2B'_i\beta'+2C'_i\gamma' = 0 \ldots\ldots\ldots(3);$$
and the problem 4 is the geometrical equivalent of the analytical problem, "To determine the ratios of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ from five linear and independent equations of the type (3)." In these equations the coefficients may have any values whatever; whereas, in the problem of M. de Jonquières, the five equations are subject to the condition, that in each of them the symmetrical determinant formed with the six coefficients must be equal to zero; and in the problem, "to determine the conic passing through five given points," there is the still further limitation, that the first minors of those determinants must also be equal to zero.

We shall denote the systems of conics represented by the equations

(i) ..... $\lambda_1\sigma_1+\lambda_2\sigma_1 = 0,$

(ii) ..... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3 = 0,$

(iii) ..... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3+\lambda_4\sigma_4 = 0,$

(iv) ..... $\lambda_1\sigma_1+\lambda_2\sigma_2+\lambda_3\sigma_3+\lambda_4\sigma_4+\lambda_5\sigma_5 = 0,$

by the symbols (σ_1, σ_2), $(\sigma_1, \sigma_2, \sigma_3)$, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and $(\sigma_1, \sigma_3, \sigma_3, \sigma_4, \sigma_5)$; and we shall describe them as *systems of the orders* 1, 2, 3, 4 respectively. The coefficients λ are absolutely indeterminate, and it is understood that the conics σ are *independent*, *i.e.* that σ_3 does not belong to the system (σ_1, σ_2), nor σ_4 to the system $(\sigma_1, \sigma_2, \sigma_3)$, nor σ_5 to the system $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. If the equations (i), (ii), (iii), (iv), are in line-coordinates, we shall describe the corresponding systems of conics as *tangential systems of the orders* 1, 2, 3, and 4. In the enunciations and solutions of the problems 1—4, we have tacitly supposed that the data are such as to render them determinate; the necessary and sufficient condition for this determinateness is that the five conics S_1, S_2, S_3, S_4, S_5, (or the pairs of points, real, imaginary, or coincident, by which any, or all of them, are replaced) should be independent, or should form a tangential system of order 4. A system of order 1 is the "faisceau," a system of order 2 is the "réseau," of French geometers. A single conic may be regarded as forming a system of order 0.

It is evident that a system of order k is determined by $k+1$ inde-

pendent conics, and that if $k+1$ independent conics of a system of order k harmonically circumscribe a given conic, every conic of the system harmonically circumscribes that conic. (The geometric proof for the case $k=1$, which has been given above, might easily be extended to the other cases). We have also the important theorem:

"All the conics of a given system of order k circumscribe harmonically all the conics of a certain tangential system of order $4-k$; and, conversely, all the conics which circumscribe harmonically the conics of a given tangential system of $4-k$, form a system of order k."

The tangential system of order $4-k$, which thus corresponds to a given system of order k, we shall call *the system contravariant to the given system.* The relations between the two systems may be inferred from the known properties of indeterminate systems of linear equations. Thus, given $k+1$ independent conics σ ($k \gtreqless 0$, $k \leqq 4$), we have a system of $k+1$ indeterminate equations of the type,

$$\alpha_i A+\beta_i B+\gamma_i C+2\alpha'_i A'+2\beta'_i B'+2\gamma'_i C'=0 \ldots\ldots (4),$$
$$[i=1, 2, \ldots k+1,]$$

in which $\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i$ are the given coefficients, and A, B, C, A′, B′, C′ the indeterminates. The order of indeterminateness of the system is $5-k$; and if

$$A_j, B_j, C_j, A'_j, B'_j, C'_j, \ldots [j=1, 2, \ldots 5-k,] \ldots\ldots (5)$$

represent $5-k$ independent solutions, every solution is included in the formulæ

$$\Sigma . \lambda_j A_j, \quad \Sigma . \lambda_j B_j, \quad \Sigma . \lambda_j C_j, \quad \Sigma . \lambda_j A'_j, \quad \Sigma . \lambda_j B'_j, \quad \Sigma . \lambda_j C'_j,$$

the $5-k$ coefficients λ being absolutely indeterminate. This establishes the first part of the theorem; the second is the correlative, as well as the converse, of the first. Again, considering the matrix of the system (4), and the matrix of the system (5) of independent solutions as two complementary matrices, we know (Phil. Trans., vol. 151, p. 301) that the determinants of the one are proportional to the complementary determinants of the other; so that, in particular, if any determinant of either matrix is zero, the complementary determinant of the other matrix is also zero. As, perhaps, no geometrical application has hitherto been given of this analytical property, we shall refer to a few of its consequences here, though the results are such as might be obtained by simple geometrical reasoning.

(*a*). Let $k=4$; the system is determined by five independent conics, and the contravariant system is of order 0, *i.e.*, it is the single conic S harmonically inscribed in these five conics. The tangents of this conic are the conics of the given system, which degenerate into pairs of coincident straight lines; for if the line $\eta=0$, $\zeta=0$, be a tangent of S, we must have $A=0$; and therefore the complementary determinant in the matrix of the given system is also zero; *i.e.*, $x^2=0$ is one of the conics contained in the given system. Similarly, we may show that those pairs

of straight lines which are degenerate conics of the given system are no other than the pairs of straight lines which are harmonically conjugate with respect to S; viz., if the lines $(\xi = 0,\ \eta = 0)$, $(\xi = 0,\ \zeta = 0)$ are conjugate with respect to S, we must have $A' = 0$, a condition which implies that $yz = 0$ is one of the conics of the given system.

(*b*). Let $k = 3$; the given system is determined by four independent conics, the contravariant system (S_1, S_2) being a system of conics inscribed in the same quadrilateral. As in the former case, we may show that the only conics of the given system which resolve themselves into two straight lines are represented by the four common tangents of the system (S_1, S_2); and that the conics which degenerate into two straight lines are the pairs of lines harmonically conjugate with respect to the same system. Further, the three pairs of opposite vertices of the quadrilateral circumscribing the system (S_1, S_2) are conics of that system, and therefore are conics harmonically inscribed in the conics of the given system; *i.e.*, the three diagonals of that quadrilateral cut the given system in involutions of which the double points upon each diagonal are the opposite vertices of the quadrilateral on that diagonal. (Mathematical Questions from the *Educational Times*, Vol. IV., p. 110; M. Cremona, in Crelle's Journal, Vol. LXI., p. 110.)

(*c*). Let $k = 2$, so that the given system is a "réseau," and the contravariant system a tangential "réseau." As it is necessary to consider two contravariant "réseaux" in the theory of cubic curves, we shall place here the solution of two elementary problems relating to them.

Problem 5.—"Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, to determine the contravariant system."

We may determine as many elements as we please of a conic touching two given lines, and harmonically inscribed in $\sigma_1\sigma_2\sigma_3$ (Problem 2). This conic is one of the conics of the contravariant system.

Problem 6.—"To determine the conic of the system $(\sigma_1, \sigma_2, \sigma_3)$ which harmonically circumscribes two given conics."

Determine three independent conics of the contravariant system; the conic circumscribing them and the two given conics is the conic required; but if the three contravariant conics, and the two given conics, are not independent, when considered tangentially, the problem is indeterminate.

Two particular cases of the problem are of frequent occurrence:

"To determine the conic of the given system which passes through two given points;" and

"To determine the conic of the given system with regard to which a given point and line are pole and polar."

In the latter case (as indeed in the general case) only the polar system of the required conic can be obtained linearly, and the conic itself may be imaginary.

The "double points" of the given system $(\sigma_1, \sigma_2, \sigma_3)$ lie on a cubic curve, the Jacobian locus, or Hessian of the system. Each point of a pair of points self-conjugate with respect to the system is a double point of the system; and the three vertices of any triangle self-conjugate with respect to two conics of the system are three double points, of which the conjugates lie in the axis of homology of the triangle and of its polar triangle with regard to any third independent conic of the system. The contravariant system possesses the correlative properties; and its Jacobian envelope is the Pippian, or Cayleyan, of the given system. Every common chord of two conics of the given system is a tangent of the Cayleyan; for if L be a common chord of σ_2 and σ_3, and if $\omega_1\omega_2$ be the double points of the involution determined on L by the system $(\sigma_1, \sigma_2, \sigma_3)$, $\omega_1\omega_2$ is a conic of the contravariant system, so that L is a "double line" of that system, and consequently a tangent of the Cayleyan. Thus, the Cayleyan is the involution-envelope of the given system; and, reciprocally, the Hessian is the involution-locus of the contravariant system. These well known properties are introduced here to show the importance of considering explicitly the contravariant system, as the relation between the two systems of conics may be said to be the source of the contravariant relation of the Hessian and Cayleyan.

Art. 5. *Problem* 7.—"Given two conics σ_1 and σ_2, to determine the conic of the system (σ_1, σ_2) which harmonically circumscribes a given conic."

Solution.—Determine four independent conics harmonically inscribed in σ_1 and σ_2; the conic harmonically circumscribing these conics and the given conic, is the conic required. For the four auxiliary conics it will be convenient to take four pairs of points reciprocal with regard to the system (σ_1, σ_2).

Problem 8.—"Given two conics σ_1, σ_2, and two straight lines L_1, L_2; to determine the conics of the system (σ_1, σ_2) with regard to which L_1, L_2 are a pair of conjugate lines." The problem is quadratic.

Solution.—Let λ_1 be the conic reciprocal to L_1* with regard to the system (σ_1, σ_2), and let λ_1 cut L_2 in the points a, b. The conics required are the two conics A and B, with regard to which the poles of L_1 are a and b. When these points have been determined by a quadratic construction, the polar systems of the two conics will be known. But this quadratic determination we shall not require; and the fol-

* The point reciprocal to a given point P with regard to a system of conics (σ_1, σ_2) is the point in which the polars of P with regard to that system intersect. The conic reciprocal to a line is the locus of points reciprocal to the points of the line. Every such reciprocal conic passes through the three vertices of the *harmonic triangle* of the system; *i.e.*, of the triangle self-conjugate with regard to all the conics of the system.

lowing construction, which is linear, will suffice. Let X be any point on L_1, x the point reciprocal to X. The polars of X with regard to the conics (σ_1, σ_2) form a pencil of lines at x, of which the rays correspond anharmonically to the conics themselves; in this pencil xa, xb are the rays corresponding to A and B. And since the involution of λ_1 upon L_2 (of which a, b are the double points) may be obtained linearly, we can determine linearly a pencil in involution at x, of which the double rays are the rays corresponding to the conics A and B.

Problem 9.—"Given three conics σ_1, σ_2, S; to determine the conics of the system (σ_1, σ_2) which are harmonically inscribed in S." This problem depends upon the preceding, which is a particular case of it; it is of course quadratic.

Solution.—If any single element of S is given, let p, q be any two points of S, L_1 the line joining them, P, Q the points reciprocal to p, q with regard to the system (σ_1, σ_2). Let Σ be the conic reciprocal to P, Q; Σ will pass through p, q, and L_1 will be one of the chords of intersection of S and Σ; the opposite chord of intersection can then be determined linearly; let it be L_2; the two conics of the system (σ_1, σ_2) with regard to which L_1, L_2 are conjugate lines (Problem 8), are harmonically inscribed in S. For, if A be either of those conics, A is harmonically circumscribed by Σ, because Σ circumscribes the harmonic triangle of the system (σ_1, σ_2); but A is also harmonically circumscribed by the degenerate conic $L_1 L_2$; therefore A is harmonically circumscribed by S, which is a conic of the system $(\Sigma_1, L_1 L_2)$.

If only the polar system of S is given, let I be the involution of S upon any line L_1, and let λ_1 be the conic reciprocal to L_1 with regard to (σ_1, σ_2). To the involution I upon L_1 there will correspond an involution of points upon the reciprocal conic λ_1; let PQ be the polar line of the involution upon λ_1, and let Σ be the conic reciprocal to PQ; then, as before, L_1 will be one of the common chords of S and Σ, and the opposite common chord can be determined linearly.

Problem 10.—"Given three conics σ_1, σ_2, S, to find the fourth point common to the conics, which circumscribe the harmonic triangle of σ_1, σ_2, and which also harmonically circumscribe S." The harmonic triangle of (σ_1, σ_2) must not also be a self-conjugate triangle of S.

Solution.—Let P be any point in the plane of the conics, L the polar of P with regard to S; p the point reciprocal to P, and λ the conic reciprocal to L, with regard to the system (σ_1, σ_2). To the involution of S upon L, there corresponds an involution upon the conic λ; let q be the pole of this involution; the conic reciprocal to pq will circumscribe S harmonically, for it will pass through P, and will cut L in two points, which will be reciprocal to a pair of points of the involution upon λ, and which will therefore be a pair of points of the involution of S upon L. Let $p'q'$ be a second line, of which the reciprocal conic harmoni-

cally circumscribes S; let θ be the point of intersection of pq, $p'q'$; the point reciprocal to θ is the fourth point of intersection required.

This fourth point is evidently (Theorem IV., Art. 1) the centre of homology of the harmonic triangle of (σ_1, σ_2) and of its polar triangle with regard to S. We may obtain the axis of homology either by the correlative construction, or by observing that it is the polar, with regard to S, of the centre of homology.

The solution of the problem, "To determine the conic which circumscribes the harmonic triangle of (σ_1, σ_2), passes through a given point, and harmonically circumscribes a given conic," is explicitly contained in what precedes. To determine the conic which circumscribes the harmonic triangle of (σ_1, σ_2), and also harmonically circumscribes two given conics S_1, S_2, we should have to substitute successively S_1 and S_2 for S in the preceding solution, and to determine the two corresponding positions θ_1, θ_2 of the point θ; the conic reciprocal to $\theta_1\theta_2$ would then be the conic required. Though no single element of this conic is given, yet an *uneven* number of its points (the three vertices of the harmonic triangle) are given symmetrically. This explains why the conic is necessarily real, and why we can determine points on it linearly.

Art. 6.—The polar conics of a cubic curve form a system of conics of order 2. Conversely, every system of conics of order 2 may be regarded as the polar system of a certain cubic curve, which we shall call the *fundamental cubic* of the system. Let $\Delta_i = x_i \frac{d}{dx} + y_i \frac{d}{dy} + z_i \frac{d}{dz}$, $[i = 1, 2, 3]$; the analytical determination of the equation $C=0$ of the cubic curve, of which three given independent conics $\sigma_1, \sigma_2, \sigma_3$ are polar conics, requires the determination of the coordinates x_i, y_i, z_i of the poles of the three given conics. Nine equations, determining the ratios of these nine coordinates, are obtained by equating to zero the coefficients of x, y, z in the expressions

$$\Delta_2\sigma_3 - \Delta_3\sigma_2, \quad \Delta_3\sigma_1 - \Delta_1\sigma_3, \quad \Delta_1\sigma_2 - \Delta_2\sigma_1.$$

The matrix of these nine equations is skew-symmetrical; the determinant is therefore zero, and the equations can be satisfied by at least one system of ratios of the nine coordinates; and, except in special cases, by only one such system. When values have been assigned to the nine coordinates, the coefficients of C may be ascertained from the equations,

$$\Delta_1 C = \sigma_1, \quad \Delta_2 C = \sigma_2, \quad \Delta_3 C = \sigma_3.$$

There are thus two curves of the third order, and two curves of the third class, which we have to consider in connexion with a given system of conics of order 2:—(1), the fundamental cubic; (2), the Hessian; (3), the fundamental contravariant, *i. e.*, the curve of the third class related to the contravariant system of conics, precisely as

the fundamental cubic is related to the given system; (4), the Cayleyan, which, as we have seen, is related to the contravariant system as the Hessian to the given system. If the equation of the fundamental cubic is

$$x^3+y^3+z^3+6mxyz=0,$$

the equation of the Hessian is

$$m^2(x^3+y^3+z^3)-(1+2m^3)\,xyz=0;$$

the equation of the fundamental contravariant in line coordinates is

$$m(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta=0,$$

and the equation of the Cayleyan in line coordinates is

$$-m(\xi^3+\eta^3+\zeta^3)+(-1+4m^3)\,\xi\eta\zeta=0.$$

The fundamental contravariant is mentioned by Professor Cayley (*Phil. Trans.*, vol. 147, p. 427) as a curve of the third class of which the Cayleyan is the Hessian envelope. It is also the curve designated as K_3 by M. Cremona (*Introduzione ad una Teoria Geometrica delle Curve Piane*, p. 117). It may be described as the evectant of $\frac{S^3}{T^2}$, S and T being the invariants of M. Aronhold. The harmonic relation between the polar conics of the fundamental cubic and the polar conics of the fundamental contravariant may be immediately verified by means of their equations.

Art. 7. *Problem* 11.—"Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ to determine the polar systems of the fundamental cubic and of the fundamental contravariant." It will be convenient to exclude the exceptional case in which the three given conics have a pole and polar in common.

Solution.—The conics of the system (σ_2, σ_3) are, of course, all conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, and their poles all lie on a right line L_1. Let *abc* be the harmonic triangle of (σ_2, σ_3), $a'b'c'$ the polar triangle of *abc* with regard to σ_1; A, B, C the pairs of common chords of σ_1 and σ_2, which intersect at *a*, *b*, *c* respectively. By a known property of cubic curves, the polar conic of one of two points, which are self-conjugate with regard to every conic of the polar system, is the degenerate conic of that system, which consists of two straight lines intersecting at the other of the two points. Thus the poles of the conics A, B, C are respectively the intersections $(bc, b'c')$, $(ac, a'c')$, $(ab, a'b')$. Hence L_1, the locus of the poles of the system (σ_2, σ_3), which includes the conics A, B, C, is the axis of homology of the triangles *abc*, $a'b'c'$, and can be determined linearly (Problem 10). Similarly, let L_2, L_3 be the loci of the poles of the conics (σ_3, σ_1), (σ_1, σ_2); the vertices P_1, P_2, P_3 of the triangle $L_1L_2L_3$ will be respectively the poles of the conics σ_1, σ_2, σ_3.

Let P be a given point in the plane, and let σ be the polar conic of P.

If p_1 be the polar line of P with regard to σ_1, p_1 is also the polar line of P_1 with regard to σ. Thus the polar system of σ is determined linearly (Problem 6). Again, if σ be given, and its pole P be required, let p_1 be the polar of P_1 with regard to σ; the pole of p_1 with regard to σ_1 is P. Thus, in the preceding construction, it will suffice to determine the two axes of homology L_2 and L_3, since, when the pole of one conic σ_1 is known, the pole of every other conic of the system is known also.

To obtain the polar system of the fundamental contravariant, we have only to determine three conics of the contravariant system (Problem 5), and to apply to them the correlative of the preceding construction.

Art. 8. *Problem* 12. — "Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, to determine the polar system of the Hessian."

The solution of this problem depends on the following propositions:—

(1.) "The locus Σ of the poles of those conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ which are harmonically inscribed in a given conic S, is a conic."

For consider any straight line L; let (λ_1, λ_2) be the system of conics (contained in the given system) of the poles of which L is the locus; then two points of the locus Σ, and only two, lie upon L; viz., the poles of the two conics which belong to the system (λ_1, λ_2), and are harmonically inscribed in S (Problem 9). Or, we may prove the same thing analytically; for if $(x_i y_i z_i)$ is the pole of σ_i, a polar conic of the fundamental cubic C, the coefficients of σ_2 are linear in $x_i y_i z_i$; and the condition which expresses that σ_i is harmonically inscribed in S, will be quadratic in $x_i y_i z_i$; *i.e.*, the locus of $x_i y_i z_i$ is a conic section.

(2.) "Given three independent conics of the system $(\sigma_1, \sigma_2, \sigma_3)$, and the conic S, to determine Σ."

Let K be any straight line, (κ_1, κ_2) the system of conics, contained in the system $(\sigma_1, \sigma_2, \sigma_3)$, of which the poles lie on K. The poles of the conics $(\sigma_1, \sigma_2, \sigma_3)$ correspond correlatively to the polars of a fixed point X with regard to the conics themselves; for these points and lines are poles and polars with regard to the polar conic of X. If, in the construction of Problem 9, we draw the arbitrary straight line L_1 through the point X, we shall obtain a pencil in involution at the point x reciprocal to X with regard to the system (κ_1, κ_2), of which the double lines are precisely the lines corresponding to the two conics of the system (κ_1, κ_2) which are harmonically inscribed in S. The correlative involution will be an involution of points on the line K, for K is the polar of x with regard to the polar conic of X; and this involution will be no other than the involution of Σ upon K, because its double points will be the poles of the two conics (κ_1, κ_2) which are harmonically inscribed in S.

(3.) *Theorem.*—"If σ be any conic of the system $(\sigma_1, \sigma_2, \sigma_3)$, and P its

pole, the locus of the poles of the conics of the system, which are harmonically inscribed in σ, is Σ, the Hessian polar conic of P."

This theorem is easily verified analytically. For the same equation which expresses that xyz is a point of the polar conic of $x'y'z'$ with regard to the Hessian, also expresses that the polar conic of xyz with regard to the fundamental cubic is harmonically inscribed in the polar conic of $x'y'z'$ with regard to the same curve. But the theorem may also be inferred geometrically from a known property of curves of the third order. For the six points R, in which Σ cuts the Hessian, are the points conjugate (upon the Hessian) to the six points r in which σ cuts the Hessian. (M. Cremona, loc. cit. p. 108.) The six degenerate polar conics which are composed of pairs of straight lines intersecting at the points r, are to be considered as conics harmonically inscribed in σ; their poles are the six points R. Therefore the locus of the poles of the conics of the system $(\sigma_1, \sigma_2, \sigma_3)$ which are harmonically inscribed in σ, passes through the six points R; *i.e.*, that locus coincides with the Hessian polar conic of R.

(4). *Theorem.*—"If σ be any conic of the system $(\sigma_1, \sigma_2, \sigma_3)$, and P its pole, the locus of the poles of those conics of the system which harmonically circumscribe σ, is the Hessian polar line of P."

This theorem may be established by the same analysis as the last. Or it may be deduced from it geometrically; for, if σ' is any conic of the system which harmonically circumscribes σ, and if P′ is the pole of σ', P lies on the Hessian polar conic of P′, because σ is harmonically inscribed in σ'; *i.e.*, P′ lies on the Hessian polar line of P′."

It is evident that these propositions determine linearly the Hessian polar system. The polar system of the Cayleyan may be obtained by a correlative construction.

Art. 9. If C be a cubic, and Γ its Hessian, the cubics $C+\lambda\Gamma$ are termed the syzygetic cubics of C. If P is any point, σ the polar conic of P, and Σ its Hessian polar conic; its polar conic with regard to the syzygetic $C+\lambda\Gamma$ is $\sigma+\lambda\Sigma$. Thus the polar conics of a fixed point correspond anharmonically to the syzygetic cubics. Let P′, σ', Σ' represent a second given point, and its two polar conics; the reciprocal point of P′, with regard to the system (σ, Σ), will be the same as the reciprocal point of P with regard to the system (σ', Σ'). Let this point be Ω, and let Ωx be the mixed derivative of P, P′, with regard to the syzygetic $C+\lambda\Gamma$; the lines Ωx will correspond anharmonically to the polar conics of P, or of P′, and therefore to the syzygetic cubics. If we suppose that the polar systems of C and Γ are both given, the polar system of the syzygetic $C+\lambda\Gamma$, corresponding to any given line Ωx, is also given. For $\sigma+\lambda\Sigma$, the polar conic of P with regard to $C+\lambda\Gamma$, is given, since it is the conic of the system (σ, Σ) with regard to which

P′ and Ωx are pole and polar. And, similarly, if Q be any point in the plane, the polar conic of Q with regard to $C+\lambda\Gamma$ is given; for it belongs to a given system of order 1, and the polar line of P with regard to it is known, being the same as the polar line of Q with regard to $\sigma+\lambda\Sigma$.

There are three cubics of which any given cubic is the Hessian, and these three cubics are syzygetic with the given cubic. We proceed to determine the polar systems of these three cubics.

Problem 13.—"To determine the polar systems of the cubics of which a given cubic C is the Hessian." By a given cubic, we understand a cubic of which the polar system is given. The problem is, of course, cubical.

Solution.—Determine the polar system of Γ, the Hessian of C. Let P, P′ be any two points of which the second lies on the polar conic of the first; and, as before, let σ, σ' be the polar conics; Σ, Σ' the Hessian polar conics, of P, P′; Ωx the mixed derivative of P, P′ with regard to $C+\lambda\Gamma$. Consider the lines Ωx as corresponding anharmonically to the conics $\sigma'+\lambda\Sigma'$, and $\sigma+\lambda\Sigma$. For any given conic $\sigma+\lambda\Sigma$, determine the pencil in involution having its centre at the point Ω, of which the double rays Ωy_1, Ωy_2 correspond to the two conics of the system (σ', Σ'), which are inscribed harmonically in $\sigma+\lambda\Sigma$; we may obtain this determination by taking for L, in the construction of Problem 9, a straight line passing through P which is the point reciprocal to Ω with regard to the system (σ', Σ'). We thus have, at the point Ω, a pencil of lines Ωx, and a pencil of pairs of lines Ωy_1, Ωy_2. These two pencils correspond anharmonically to one another; for to each ray Ωx there corresponds but one pair Ωy_1, Ωy_2; and to each ray Ωy there corresponds but one ray Ωx, because there is but one conic in the system (σ, Σ), which circumscribes harmonically a given conic in the system (σ', Σ'). The ray Ωx will in three different directions coincide with one of the corresponding rays Ωy. Let Ωx_1, Ωx_2, Ωx_3 be the three directions of coincidence, which are to be determined by the cubic construction of M. Chasles.* The three syzygetic cubics corresponding to the three rays Ωx_2, Ωx_2, Ωx_3, are the three cubics of which C is the Hessian; because, for each of those three syzygetics, the Hessian polar conic of P passes through P′, and therefore coincides with σ; so that the Hessian itself coincides with C.

The problem, "To determine the two cubics which have the same Hessian as a given cubic C," may be solved in the same manner, but is only quadratic. In fact, when one of the three directions of coincidence Ωx_1, Ωx_2, Ωx_3, is known *à priori*, a pencil in involution of which the other two are the double lines may be determined linearly.

* Comptes Rendus, vol. 41, p. 681.

Art. 10. *Problem* 14.—"Given the polar system of a cubic C, to determine its nine points of inflexion." The problem requires one biquadratic, and two cubic constructions.

Solution.—The syzygetic cubics comprise four triangles; of which one is real, and one consists of one real line and a pair of conjugate imaginary lines; the two others are two imaginary triangles. The first two of these triangles will serve to determine the three real, and the six imaginary points of inflexion. Each of the four triangles, considered as a syzygetic cubic, is characterized by the property that it is its own Hessian. Retaining the notation of the last article, let Ωx be the ray corresponding to any given syzygetic $C+\lambda\Gamma$; and similarly let Ωh correspond to the Hessian of $C+\lambda\Gamma$; Ωx_1, Ωx_2, to those two syzygetics, other than $C+\lambda\Gamma$, which have its Hessian for their Hessians. We shall thus have, at the point Ω, a pencil of lines Ωh, and a pencil of triplets of lines Ωx, Ωx_1, Ωx_2. And since to every Hessian only one triplet of fundamental cubics corresponds, and to every cubic only one Hessian, the rays of the one pencil will correspond anharmonically to the triplets of the other. The ray Ωh will in four different directions coincide with one of the rays of its corresponding triplet; and the syzygetic cubics corresponding to the four directions of coincidence are precisely the four syzygetic triangles. Let K be any conic passing through Ω, and let the point in which any ray Ωx meets K for the second time be designated by x; also let ω, ω' be two given points, of which the first is, and the second is not, a point of K. The rays Ωh, and the conics $(\omega, \omega', x, x_1, x_2)$, will correspond to one another anharmonically; and the locus of the intersections of the corresponding rays and conics will be a cubic curve, which will cut K in the two points Ω, ω, and in four other points h_1, h_2, h_3, h_4, which may be determined by a biquadratic construction indicated by M. Chasles.* The four rays Ωh_1, Ωh_2, Ωh_3, Ωh_4, (of which two, and only two, are real) are the four directions of coincidence. Let Ωh be either of the two real rays. The polar system of the syzygetic triangle corresponding to Ωh, is real, and its polar conics all pass through the three vertices of the triangle. Determine two of these polar conics which pass through one and the same point taken arbitrarily in the plane; the three remaining intersections of the two conics are to be obtained by a cubic construction, and are the three vertices of the syzygetic triangle.

* Comptes Rendus, vol. 41, p. 1193.

FIFTH SESSION, 1868–9.

November 12th, 1868.

SPECIAL GENERAL MEETING, held at Burlington House.

Prof. SYLVESTER, F.R.S., President, in the Chair.

This Meeting was convened for the purpose of altering Rule 30, by omitting the words from "on the fourth Thursday" to "the second Thursday" inclusive, and in place thereof substituting the words, "on such day or days in each month, from November to June, as the Council shall from time to time determine."

After a few explanatory remarks from the President, the above-named change was proposed by Mr. Spottiswoode, V.P., seconded by Prof. W. G. Adams, and carried unanimously.

The business of this Meeting being then concluded, there was held

THE ANNUAL GENERAL MEETING.

The Reports of the Treasurer and Secretaries were read and adopted.

From the Report of the Secretaries, it appeared that the number of Members, since the last General Meeting, held November 14th, 1867, had increased from 104 to 115.

The communications made to the Society during the past Session had been as follows:—

"On some Theorems of a new character in Local Probability:" by Mr. Crofton.

"On the representation of Points in Space by Triplets of Points on a Line:" by Prof. Hirst.

"A short account of a System of Coordinates of Lines in Space:" by Mr. Spottiswoode.

"On the Logarithms of Imaginary Quantities;" and "On the Logarithmic Relations of the Double Tangents of a Quartic Curve:" by Prof. Cayley.

"On a Law of Periodicity in the derivation of point from point on Cubic Curves, and its connection with the Theory of Allineation:" by Prof. Sylvester.

"On the construction of Stereograms of Surfaces;" and "On the doctrine of Reciprocal Diagrams of Forces, with the extension of Airy's Function of Stress from two dimensions to three:" by Prof. Clerk-Maxwell.

"On the Anharmonic-Ratio Sextic:" by Mr. J. J. Walker.

"On the Equilibrium of Forces in Space:" by Mr. Spottiswoode.

"On Circles and Spheres:" by Mr. Clifford.

"The Construction of the last point of Intersection of a Cubic Curve by a Curve of superior order;" and "A Geometrical Note on the Concomitants of a Binary Cubic:" by Prof. H. J. S. Smith.

"On a Mode of Generation of a Sibi-Reciprocal Surface:" by Prof. Cayley.

"An Account of a Method for the Inscription of Closed Polygons in any Quadric Surface:" by Mr. M. Gardiner, C.E.

"On the eight Points of Intersection of three Quadric Surfaces:" by Mr. T. Cotterill.

"On the Theory of certain Systems of Conics which present themselves in connection with Cubic Curves:" by Prof. H. J. S. Smith.

"A Solution of Lexell's Theorem:" by the Lord Bishop of Limerick; communicated by Prof. Sylvester.

"On a certain Double Integral:" by Mr. Crofton.

"Euclid as a Text-Book of Elementary Geometry:" by Mr. J. M. Wilson.

"A Problem in Combinations:" by Prince Camille de Polignac.

"On Functional Evolution:" by Mr. W. S. B. Woolhouse.

Many lesser communications were made by Prof. Sylvester, Prof. Brayley, Messrs. Cotterill, S. Roberts, and Walker.

The same Mathematical Journals had been subscribed for as in the preceding Session, except that the "Philosophical Magazine" had been taken in the place of Crelle's Journal, which the Society receives as a present from Herr Borchardt.

The Proceedings of the Society had been sent to the Royal Societies of London and of Edinburgh; the Royal Irish Academy; the Cambridge Philosophical Society; the Philosophical Society of Manchester; the Institute of Actuaries; the British Museum; l'Académie des Sciences, Paris; l'École Polytechnique; la Société Philomathique; Akademie der Wissenschaften, Berlin; il Istituto Lombardo, Milan; Prof. Luigi Cremona, an Editor of the Annali di Matematica; Herr Borchardt, Berlin, Editor of Crelle's Journal.

In return for these, the Society had received the Proceedings of the Royal Society; the Journal of the Institute of Actuaries; Memorie and Rendiconti of the Istituto Lombardo; the Annali di Matematica; the Monatsberichte for 1868, from the Akademie der Wissenschaften; Crelle's Journal, from Herr Borchardt.

The following presents had been made to the Society:—

"On Centres, Faisceaux, and Envelopes of Homology;" and "A Note on Prof. Tait's 'Quaternion-Path' to Determinants of the Third Order:" by the Rev. Hugh Martin.

"A Note on the Importance of the Spectroscopical Examinations of the Vicinity of the Sun when totally Eclipsed, &c.:" by Prof. Brayley.

"Un Teorema intorno alle Forme Quadratiche non omogenie fra due Variabili:" by Prof. Cremona.

"Preliminari di una Teoria Geometrica delle Superficie," 1860, parts 1 and 2: by Prof. Cremona.

"Sulle Trasformazioni Geometriche delle Figure Piane," 2° Note, 1865: from Prof. Cremona.

"Abhandlungen über die Mechanische Wärmetheorie, von R. Clausius:" from Prof. Hirst.

"On the Cyclide;" and "On the Equilibrium of a Spherical Envelope:" from Mr. J. Clerk-Maxwell.

Hargreave's "Essay on the Resolution of Numerical Equations:" from Mrs. Hargreave.

"Note sur l'Équilibre des Forces dans l'Espace:" from Mr. Spottiswoode.

"Three Supplements to Sir W. R. Hamilton's Essay on the Theory of Systems of Rays:" from Dr. Ingleby.

"Besant's Hydrostatics and Hydrodynamics;" and "Rapport fait à l'Académie Royale des Sciences des Pays Bas:" from Prof. Hirst.

"Catalogue of Scientific Papers, vol. i., A—CLU:" from the Royal Society.

"*Comptes Rendus.* Rapport sur un Mémoire de M. Mannheim sur le Déplacement d'une Figure de Forme Invariable" (Rapporteur, M. Chasles): from Prof. Mannheim.

"A Translation of the First Book of Laplace's Mécanique Céleste, by the Rev. John Toplis, M.A.:" from Dr. Ingleby.

The following Officers and Council were elected to serve during the ensuing year:—President, Prof. Cayley; Vice-Presidents, Mr. J. Clerk-Maxwell and Profs. H. J. S. Smith and Sylvester; Treasurer, Prof. Hirst; Secretaries, Messrs. Jenkins and Tucker; other Members of Council, Messrs. Clifford, T. Cotterill, Crofton, A. De Morgan, Roberts, Spottiswoode, Stirling, A. Smith, and Woolhouse.

Prof. Cayley, the newly-elected President, then took the chair. The Lord Bishop of Limerick was elected a Member of the Society.

Votes of thanks were carried to the Chemical Society for the renewed use of their rooms; to Dr. Sharpey for the continued use of shelves in his room at University College; and to the retiring Members of the Council for their services.

After the conclusion of the business of the Meeting,

Professor Cayley gave an account to the Meeting of a Memoir by Herr Listing, "Census räumlicher Complexe oder Verallgemeinerung des Euler'schen Satzes von den Polyedern," published in the Göttingen

Transactions for 1862. The fundamental theorem is a relation $a-(b-\kappa)+(c-\kappa'+\pi)-(d-\kappa''+\pi'-\omega)=0$ existing in any figure whatever between a the number of points, b the number of lines, c the number of areas, d the number of spaces, and certain supplementary quantities $\kappa, \kappa', \kappa'', \pi, \pi', \omega$. In an extensive class of figures these last are each $=0$, and the relation is $a-b+c-d=0$; thus, in a closed box, $a=8$, $b=12$, $c=6$, $d=2$ (viz., there is the finite space inside, and the infinite space outside, the box): if the box be opened, $a=10$, $b=15$, $c=6$, $d=1$; if the lid be taken away, $a=8$, $b=12$, $c=5$, $d=1$; in each case, $a-b+c-d=0$. If the bottom be also taken away, $a=8$, $b=12$, $c=4$, $d=1$; but here one of the supplementary quantities comes in, $\kappa''=1$, and the theorem is $a-b+c-(d-\kappa'')=0$. The chief difficulty and interest of the Memoir consist in the determination of the supplementary quantities $\kappa, \kappa', \kappa'', \pi, \pi', \omega$.

Dr. Henrici then read the following paper:—

"*On certain formulæ concerning the Theory of Discriminants; with applications to Discriminants of Discriminants, and to the Theory of Polar Curves.*" By Olaus M. F. E. Henrici, Ph.D.

Let $$u=(a', a'', a''', \ldots)(y_1, y_2, y_3 \ldots)^m$$

be a quantic in the variables $y_1, y_2, y_3, \ldots$ and let Ξ be its discriminant. Ξ is a function of the coefficients $a', a'', a''' \ldots$. If these coefficients are related by the equation $\Xi=0$, then $u=0$ has a system of double roots, which I denote by $\eta_1, \eta_2, \eta_3 \ldots$. For these values of Ξ and of the variables y, the proportion holds

$$(1) \qquad \frac{\delta u}{\delta a'} : \frac{\delta u}{\delta a''} : \frac{\delta u}{\delta a'''} : \ldots = \frac{\delta \Xi}{\delta a'} : \frac{\delta \Xi}{\delta a''} : \frac{\delta \Xi}{\delta a'''} : \ldots,$$

which is used to determine the singular roots η, as the $\frac{\delta u}{\delta \eta}$ are products of powers of the η's. If the coefficients a in u, altogether or some of them, be any functions whatsoever of one or more new variables $x_1, x_2, x_3, \ldots$, then Ξ becomes a function of these x's. Dr. Salmon has proved, in the second edition of his "Lessons introductory to the modern higher Algebra," that the above proportion is true also for the differential coefficients of Ξ with respect to these x's. Or we have

$$(2) \qquad \frac{\delta u}{\delta x_1} : \frac{\delta u}{\delta x_2} : \frac{\delta u}{\delta x_3} : \ldots\ldots = \frac{\delta \Xi}{\delta x_1} : \frac{\delta \Xi}{\delta x_2} : \frac{\delta \Xi}{\delta x_3} : \ldots\ldots$$

The only condition that both proportions may hold is that Ξ should vanish, and that the variables y should be replaced by the singular roots η. Hence both proportions hold simultaneously, and we may combine

them into one. Denoting by a any one of the coefficients $a', a'', a''', \ldots$, and by x any one of the variables $x_1, x_2 \ldots$, we obtain thus

$$\frac{\delta u}{\delta a} : \frac{\delta u}{\delta x} = \frac{\delta \Xi}{\delta a} : \frac{\delta \Xi}{\delta x},$$

or, as I prefer to write it,

$$\frac{\delta u}{\delta a}\frac{\delta \Xi}{\delta x} = \frac{\delta \Xi}{\delta a}\frac{\delta u}{\delta x}. \tag{3}$$

The proportion (2) is in fact a simple consequence of the first. Ξ contains x implicitly only in the a's; hence we have for the partial differential coefficient of Ξ with respect to x,

$$\frac{\delta \Xi}{\delta x} = \frac{\delta \Xi}{\delta a'}\frac{\delta a'}{\delta x} + \frac{\delta \Xi}{\delta a''}\frac{\delta a''}{\delta x} + \frac{\delta \Xi}{\delta a'''}\frac{\delta a'''}{\delta x} + \ldots;$$

and by aid of (1),

$$\frac{\delta u}{\delta a}\frac{\delta \Xi}{\delta x} = \frac{\delta \Xi}{\delta a}\left(\frac{\delta u}{\delta a'}\frac{\delta a'}{\delta x} + \frac{\delta u}{\delta a''}\frac{\delta a''}{\delta x} + \frac{\delta u}{\delta a'''}\frac{\delta a''}{\delta x} + \ldots\right)$$
$$= \frac{\delta \Xi}{\delta a}\frac{\delta u}{\delta x},$$

which is (3). It was in this way that I first arrived at the formula (3), and it is not without interest to see (3) proved by (1), as it shows that (3) holds for *any* function of the coefficients $a', a'', a''' \ldots$, for which the proportion (1) holds. This is the case, for instance, with the first differentials $\frac{\delta \Xi}{\delta a}$ as soon as $\eta_1, \eta_2, \eta_3 \ldots$ represent not merely a system of double roots but of treble roots of $u = 0$.

The formula (3) is very useful in a great number of problems, which depend upon discriminants involving one or more variables. The factor $\frac{\delta u}{\delta a}$ on the left-hand side is a product of powers of the singular roots $\eta_1, \eta_2, \eta_3, \ldots$ and therefore in general does not vanish. We can at least always find one coefficient a for which $\frac{\delta u}{\delta a}$ is not $= 0$, as all the variables do not vanish simultaneously; or we may transform our quantic u by linear transformation in such a way, that none of the singular roots vanish. I shall therefore always suppose $\frac{\delta u}{\delta a}$ not to be $= 0$. Then (3) shows that $\frac{\delta \Xi}{\delta x}$ vanishes in two cases, either if $\frac{\delta \Xi}{\delta a} = 0$, or if $\frac{\delta u}{\delta x} = 0$. To see whether in these cases $\frac{\delta^2 \Xi}{\delta x^2}$ also vanishes, it is necessary to develope some new formulæ. For shortness' sake I take a binary quantic, and suppose the coefficients functions of one variable x only, the whole proceeding being easily transferred to quantics in general.

Let $f(y,1)$, or more shortly $f(y)$, be a binary quantic

$$f(y) = a_m y^m + a_{m-1} y^{m-1} + \ldots\ldots + a_1 y + a_0,$$

and let Ξ be its discriminant. I take Ξ as the resultant of $f(y) = 0$, $f'(y) = 0$. If $y_1, y_2, \ldots y_{m-1}$ be the roots of $f'(y) = 0$, we have

$$\Xi = a_m^{m-2} f(y_1) f(y_2) \ldots f(y_{m-1}).$$

The coefficients a_k are supposed functions of x. Then the y_i's become functions of x determined by the equation $f'(y) = 0$. From this expression of Ξ we obtain

$$\Xi'(x) = a_m^{m-2} \frac{df(y_1)}{dx} f(y_2) f(y_3) \ldots + \ldots,$$

where all terms not written vanish with $f(y_1) = 0$. If now $x = \xi$ be a root of $\Xi = 0$, $y = \eta$ the corresponding double root of $f(y) = 0$ and $y_1 = \eta$, hence $f(y_1) = f(\eta) = 0$, we have

$$\Xi'(\xi) = a_m^{m-2} \frac{df(\eta)}{d\xi} f(y_2) f(y_3) \ldots\ldots$$

Now

$$\frac{df(\eta)}{d\xi} = \frac{\delta f(\eta)}{\delta \xi} + \frac{\delta f(\eta)}{\delta \eta} \frac{\delta \eta}{\delta \xi} = \frac{\delta f(\eta)}{\delta \xi},$$

as $\frac{\delta f(\eta)}{\delta \eta} = 0$. We may therefore, in the expression for $\Xi'(\xi)$, replace the total differential coefficient $\frac{df(\eta)}{d\xi}$ by the partial one $\frac{\delta f(\eta)}{\delta \xi}$.

To find a simple expression for the factor $a_m^{m-2} f(y_2) f(y_3) \ldots$ of $\Xi'(\xi)$, I suppose x remains unaltered and equals ξ. Then the coefficients a_i have constant values, which satisfy the equation $\Xi = 0$. From these I separate one, say a_k, and substitute $a_k + \sigma$ for a_k. Ξ vanishes for $\sigma = 0$, and to this value of σ corresponds again the double root $y = \eta$ of $f(y)$. Differentiating Ξ with regard to σ, we obtain, exactly as before, for x

$$\frac{\delta \Xi}{\delta \sigma} = a_m^{m-2} \frac{\delta f(\eta)}{\delta \sigma} f(y_2) f(y_3) \ldots,$$

as soon as $\sigma = 0$. But $\frac{\delta f(\eta)}{\delta \sigma} = \frac{\delta f(\eta)}{\delta a_k} = \eta^k$, and $\frac{\delta \Xi}{\delta \sigma} = \frac{\delta \Xi}{\delta a_k}$, whence

$$\Xi_k = a_m^{m-2} \frac{\delta f(\eta)}{\delta a_k} f(y_2) f(y_3) \ldots,$$

where Ξ_k is written for $\frac{\delta \Xi}{\delta a_k}$. This in combination with the above expression for $\Xi'(\xi)$ gives

$$\frac{\delta f(\eta)}{\delta a_k} \Xi'(\xi) = \Xi_k \frac{\delta f(\eta)}{\delta \xi},$$

or

$$\eta^k \Xi'(\xi) = \Xi_k \frac{\delta f(\eta)}{\delta \xi}.$$

As we suppose that $\frac{\delta f(\eta)}{\delta a_k} (= \eta^k)$ does not vanish, this formula shows

that $\Xi'(\xi)$ vanishes, or that $\Xi=0$ has a double root $x=\xi$, when either $\Xi_k=0$ or $\frac{\delta f(\eta)}{\delta\xi}=0$, and in no other case. These cases are to be separated.

1. Let $\Xi_k=0$, $\frac{\delta f(\eta)}{\delta\xi}$ not $=0$. When this is satisfied for one index k, it must be satisfied for every k, or else we must have $\frac{\delta f(\eta)}{\delta a_k}=\eta^k=0$ against our supposition. The vanishing of every Ξ_k is the condition that $f(y)$ may have two pairs of double roots η_1, η_2, or one treble root η.

Differentiating Ξ twice with respect to x, we get

$$\Xi''(x)=2a_m^{m-2}\,\frac{\delta f(y_1)}{\delta x}\,\frac{\delta f(y_2)}{\delta x}\,f(y_3)\ldots+\ldots,$$

where all terms not written vanish with $f(y_1)=0$, $f(y_2)=0$; or, if we substitute $x=\xi$, $y_1=\eta_1$, $y_2=\eta_2$, $f(y_1)=f(y_2)=0$, we have

$$\Xi''(\xi)=2a_m^{m-2}\,\frac{\delta f(\eta_1)}{\delta\xi}\,\frac{\delta f(\eta_2)}{\delta\xi}\,f(y_3)\ldots.$$

If we further substitute $a_i+\rho$ and $a_k+\sigma$ for a_i and a_k, whilst x remains $=\xi$, we find, by differentiating with respect to ρ and σ, and making $\rho=\sigma=0$,

$$\Xi_{ik}=\frac{\delta^2\Xi}{\delta a_i\,\delta a_k}=a_m^{m-2}\left(\frac{\delta f(\eta_1)}{\delta a_i}\,\frac{\delta f(\eta_2)}{\delta a_k}+\frac{\delta f(\eta_2)}{\delta a_i}\,\frac{\delta f(\eta_1)}{\delta a_k}\right)f(y_3)\ldots$$

$$=\left(\frac{\delta f(\eta_1)}{\delta a_i}\,\frac{\delta f(\eta_2)}{\delta a_k}\right)a_m^{m-2}\,f(y_3)\ldots.$$

These two expressions for $\Xi''(\xi)$ and Ξ_{ik} hold simultaneously, and may be combined into

$$\left(\frac{\delta f(\eta_1)}{\delta a_i}\,\frac{\delta f(\eta_2)}{\delta a_k}\right)\Xi''(\xi)=2\Xi_{ik}\,\frac{\delta f(\eta_1)}{\delta\xi}\,\frac{\delta f(\eta_2)}{\delta\xi},$$

which holds good for two distinct double roots η_1, η_2.

Similarly we find

$$\frac{\delta f(\eta)}{\delta a_i}\,\frac{\delta f(\eta)}{\delta a_k}\,\Xi''(\xi)=\Xi_{ik}\left(\frac{\delta f(\eta)}{\delta\xi}\right)^2$$

for a treble root. Here

$$\Xi_{ik}=\frac{\delta^2\Xi}{\delta a_i\,\delta a_k},$$

$$\left(\frac{\delta f(\eta_1)}{\delta a_i}\,\frac{\delta f(\eta_2)}{\delta a_k}\right)=\frac{\delta f(\eta_1)}{\delta a_i}\,\frac{\delta f(\eta_2)}{\delta a_k}+\frac{\delta f(\eta_1)}{\delta a_k}\,\frac{\delta f(\eta_2)}{\delta a_i}.$$

When for one value $x=\xi$ a greater number of double roots $\eta_1, \eta_2 \ldots \eta_\mu$ exists, we get

$$\left(\frac{\delta f(\eta_1)}{\delta a_{i_1}}\ \frac{\delta f(\eta_2)}{\delta a_{i_2}} \dots \frac{\delta f(\eta_\mu)}{\delta a_{i_\mu}}\right) \Xi^\mu(\xi) = 1.2 \dots \mu\, \Xi_{i_1 i_2 \dots i_\mu} \frac{\delta f(\eta_1)}{\delta \xi} \dots \frac{\delta f(\eta_\mu)}{\delta \xi},$$

where the expression within the brackets denotes the sum of all the $1.2 \dots \mu$ similar expressions, which are obtained by permutating all the η_k's. When two of these double roots η_1, η_2 coincide, forming a treble root η, this formula changes into

$$\left(\frac{\delta f(\eta)}{\delta a_{i_1}}\ \frac{\delta f(\eta)}{\delta a_{i_2}}\ \frac{\delta f(\eta_3)}{\delta a_{i_3}} \dots\right) \Xi^\mu(\xi) = 1.2 \dots \mu\, \Xi_{i_1 i_2 \dots i_\mu} \left(\frac{\delta f(\eta)}{\delta \xi}\right)^2 \frac{\delta f(\eta_3)}{\delta \xi} \dots.$$

and similarly when more roots coincide.

2. Let $\frac{\delta f(\eta)}{\delta \xi} = 0$, but Ξ_k not $= 0$. For such a double root $x = \xi$ of Ξ there exists one double root η of $f(y)$ only. Differentiating Ξ twice, we obtain

$$\Xi''(\xi) = a_m^{m-2} \frac{d^2 f(\eta)}{d\xi^2} f(y_2)\, f(y_3) \dots.$$

Here we cannot replace the total differential coefficient $\frac{d^2 f(\eta)}{d\xi^2}$ by the partial one. We have to develope $\frac{d^2 f(\eta)}{d\xi^2}$, and to substitute the value of $\frac{\delta \eta}{\delta \xi}$, which is obtained by differentiating the equation $f'(y) = 0$. The result is

$$\frac{d^2 f(\eta)}{d\xi^2} = \frac{\frac{\delta^2 f(\eta)}{\delta \xi^2}\ \frac{\delta^2 f(\eta)}{\delta \eta^2} - \left(\frac{\delta^2 f(\eta)}{\delta \xi\, \delta \eta}\right)^2}{\frac{\delta^2 f(\eta)}{\delta \eta^2}}.$$

In combination with a former equation, we thus find

$$\frac{\delta f(\eta)}{\delta a_k} \Xi''(\xi) = \Xi_k \frac{d^2 f(\eta)}{d\xi^2}$$

$$= \Xi_k \frac{\frac{\delta^2 f(\eta)}{\delta \xi^2}\ \frac{\delta^2 f(\eta)}{\delta \eta^2} - \left(\frac{\delta^2 f(\eta)}{\delta \xi\, \delta \eta}\right)^2}{\frac{\delta^2 f(\eta)}{\delta \eta^2}}.$$

The denominator $\frac{\delta^2 f(\eta)}{\delta \eta^2}$, in general, does not vanish. This would indicate a treble root η of $f(\eta) = 0$.

These different cases, in which ξ can become a double root of Ξ, are clearly distinguished by a geometrical representation of $f(y)$. If x and y are taken as coordinates of a point, $f(y) = 0$ represents a curve. $\Xi = 0$ gives the equation to all tangents $x = \xi$ parallel to the axis of y. If for one of these tangents two double roots η_1, η_2 exist, it becomes a

double tangent; and it is a tangent of inflexion in case of a treble root. In both cases the Ξ_k's vanish. But if $\frac{\delta f(\eta)}{\delta \xi} = 0$, the curve has a double point, which absorbs two tangents; and if besides $\frac{d^2 f}{d\xi^2} = 0$, this double point becomes a cusp, which absorbs three tangents, and Ξ has a treble root.

When in case of a double tangent $\frac{\delta f}{\delta \xi} = 0$ for one point of contact, this becomes a double point. We then have $\Xi''(\xi) = 0$, and

$$\left(\frac{\delta f(\eta_1)}{\delta a_i} \frac{\delta f(\eta_2)}{\delta a_k}\right) \Xi'''(\xi) = 3\,\Xi_{ik} \frac{d^2 f(\eta_1)}{d\xi^2} \frac{\delta f(\eta_2)}{\delta \xi};$$

and so on for other cases.

All that has been said about discriminants of binary functions holds for quantics in general. Assuming the coefficients to be functions of more than one variable, our formulæ hold for any one of them. Some new ones are easily added for the higher differential coefficients with respect to more variables.

It is further clear that we may substitute resultants in general for discriminants. I have treated of discriminants only, because the applications I intend to make belong to discriminants alone.

Discriminants of Discriminants.

I suppose now that in our quantic

$$f(y) = a_m y^m + a_{m-1} y^{m-1} + \ldots + a_1 y + a_0,$$

the coefficients a_k are integral functions in x all of the degree n,

$$a_k = a_n^{(k)} x^n + a_{n-1}^{(k)} x^{n-1} + \ldots + a_0^{(k)}.$$

Hence $f(y)$ may be written

$$f(y) = u_0 + x u_1 + x^2 u_2 + \ldots + x^n u_n,$$

where the u_n's are binary quantics in $(y, 1)$ of the order m, with coefficients $a_i^{(k)}$. The discriminant Ξ is (in x) of the degree $2n(m-1)$,

$$\Xi = A_0 + A_1 x + A_2 x^2 + \ldots.$$

I form the discriminant D of Ξ, and shall show that

$$D = \Delta B^2 C^3,$$

where $B = 0$, *of the order* $4n(m-2)(m-3)$ *in the* $a_i^{(k)}$*'s, is the condition that* $f(y)$ *may have for a certain value of* x *two double roots, where* $C = 0$, *of the order* $6n(m-2)$, *is the condition for a treble root of* $f(y)$, *and where* $\Delta = 0$, *of the order* $6mn - 4(m+n) + 4$, *is the condition that for a double root* η *of* $f(y)$ *the differential coefficient* $\frac{\delta f(\eta)}{\delta \xi}$ *may vanish.*

Δ therefore is the discriminant of f, treated as a function of x and y.

This theorem is, according to Dr. Salmon, given by Prof. Cayley for $n=1$, or for $f(y) = u_0 + xu_1$, but without a general proof. By aid of the formulæ before developed, the proof is very easy.

$D = 0$ is the condition that Ξ shall have a double root ξ. But we have seen that Ξ has a double root only in one of the three cases where Δ, B, or C vanishes. Hence D must contain these quantities as factors, and it cannot contain any other factor. To show that they enter as a linear, squared, or cubed factor respectively, I suppose the coefficients $a_i^{(k)}$ of $f(y)$ linear functions of a new variable λ. Then Δ, B, C become functions of λ.

Let λ be a root of $B=0$, then Ξ has a double root ξ, and $f(y)$ has two double roots η_1, η_2. For these values we have, as D is the discriminant of Ξ,

$$\frac{\delta\Xi}{\delta A_k} D'(\lambda) = \frac{\delta D}{\delta A_k} \frac{\delta\Xi}{\delta\lambda},$$

and also

$$\frac{\delta f(\eta)}{\delta a_k} \frac{\delta\Xi}{\delta\lambda} = \Xi_k \frac{\delta\Xi}{\delta\lambda}.$$

Now for two double roots η_1, η_2 of $f(y)$ we have $\Xi_k = 0$, hence $\frac{\delta\Xi}{\delta\lambda} = 0$, hence also $D'(\lambda) = 0$. Or each root λ of B is a double root of D, or B^2 *is a factor of* D.

Further, for the same values of λ, x, y,

$$\frac{\delta\Xi}{\delta A_k} D''(\lambda) = \frac{\delta D}{\delta A_k} \frac{d^2\Xi}{d\lambda^2}.$$

$D''(\lambda)$ vanishes therefore only if $\frac{d^2\Xi}{d\lambda^2}$ vanish, because $\frac{\delta D}{\delta A_k}$ in general does not vanish. This would indicate that for one value of λ two double roots ξ of Ξ exist. But

$$\frac{d^2\Xi}{d\lambda^2} = \frac{\frac{\delta^2\Xi}{\delta\xi^2}\frac{\delta^2\Xi}{\delta\lambda^2} - \left(\frac{\delta^2\Xi}{\delta\xi\,\delta\lambda}\right)^2}{\frac{\delta^2\Xi}{\delta\xi^2}},$$

or $D''(\lambda)$ vanishes together with

$$\frac{\delta^2\Xi}{\delta\xi^2}\frac{\delta^2\Xi}{\delta\lambda^2} - \left(\frac{\delta^2\Xi}{\delta\xi\,\delta\lambda}\right)^2.$$

For the same values of λ, ξ, η_1, η_2 we have

$$\left(\frac{\delta f(\eta_1)}{\delta a_i}\frac{\delta f(\eta_2)}{\delta a_k}\right)\frac{\delta^2\Xi}{\delta\xi^2} = 2\Xi_{ik}\frac{\delta f(\eta_1)}{\delta\xi}\frac{\delta f(\eta_2)}{\delta\xi},$$

and similarly

$$\left(\frac{\delta f(\eta_1)}{\delta a_i}\frac{\delta f(\eta_2)}{\delta a_k}\right)\frac{\delta^2\Xi}{\delta\xi\,\delta\lambda} = \Xi_{ik}\left(\frac{\delta f(\eta_1)}{\delta\xi}\frac{\delta f(\eta_2)}{\delta\lambda}\right),$$

$$\left(\frac{\delta f(\eta_1)}{\delta a_i}\ \frac{\delta f(\eta_2)}{\delta a_k}\right)\frac{\delta^2 \Xi}{\delta \lambda^2} = 2\Xi_{ik}\frac{\delta f(\eta_1)}{\delta \lambda}\ \frac{\delta f(\eta_2)}{\delta \lambda}.$$

Introducing these values in

$$\frac{\delta^2 \Xi}{\delta \xi^2}\ \frac{\delta^2 \Xi}{\delta \lambda^2} - \left(\frac{\delta^2 \Xi}{\delta \lambda \delta \xi}\right)^2,$$

we obtain an expression which in general does not vanish. Hence B is a squared factor only of D.

But our expression vanishes as soon as the two double roots η_1, η_2 coincide with a treble root. In this case, therefore, $D''(\lambda)$ vanishes. This λ is a root of $C = 0$. Hence each root of C is a treble root of D, or C^3 *is a factor of* D. It is easily seen that C cannot be a factor of a higher degree.

The third factor Δ of D enters only in the first degree. For we have

$$\frac{\delta \Xi}{\delta A_k} D'(\lambda) = \Xi'(\lambda)\frac{\delta D}{\delta A_k},$$

and neither $\Xi'(\lambda)$ nor $\frac{\delta D}{\delta A_k}$ vanishes necessarily when $\Delta = 0$. Therefore, as above stated, $D = \Delta . B^2 . C^3$.

To ascertain the orders of Δ, B, and C, I first determined the order of B and C by methods indicated by Professor Cayley and Dr. Salmon, and then subtracted the order of B^2C^3 from that of D, to find the order of Δ.

The quantities B, C, and Δ are of very different character. Writing $f(y)$ in the form

$$f(y) = u_0 + xu_1 + x^2u_2 + \ldots + x^nu_n,$$

B and C are simultaneous invariants of the u_k's; but Δ is an invariant of f, taken as a function of x and y. Δ therefore remains unaltered by linear transformation of x and y, whilst B and C are altered. But B and C retain their invariable character for certain transformations of higher order. If for y such a rational function of x, and a new variable y', be substituted, that y' also can be expressed as a rational function of x and y, and if for the transformed function f the quantities B', C' are formed analogous to B and C, then B'C' must vanish together with B and C respectively; and *vice versâ*. This becomes evident by the geometrical representation of $f(y)$, before mentioned; and still more so, when we make use of Riemann's peculiar representation of algebraical functions by surfaces, having as many leaves as the function has branches. Our function $f(y, x)$ is exactly of that form from which Riemann starts in his theory of Abel's functions. The substitution mentioned is of that kind which Riemann recommends for the transformation of algebraical functions into others which lead to the same class of Abel's functions. For these transformations, therefore, the invariants B and C must be of importance.

I add the remark, that the theorem about discriminants of discriminants is also true, if for the quantic $f(y)$ another is substituted containing any number of variables $y_1, y_2, y_3 \ldots$, as long as the coefficients remain functions of one variable only. The proof given holds for this case. Professor Cayley has given the theorem for binary and ternary functions.*

On Polar Curves.

Let $$v = \Sigma a_{\kappa\lambda\mu} x_1^{\kappa} x_2^{\lambda} x_3^{\mu}, \quad \kappa+\lambda+\mu = r,$$

be a ternary quantic of the order r, and let the coefficients $a_{\kappa\lambda\mu}$ be quantics with three other variables y_1, y_2, y_3 of the order m. Then $v=0$ represents as a function of the x's a system of curves of the order r, of such a nature that m^2 curves pass through any two given points. Or, all curves, $v=0$, which pass through one given point form a series of the index m^2. Such series of curves are formed by the mth polars of all points in a plane with respect to a curve of the order $n=m+r$. The y's represent in this case the coordinates of the poles. I shall use the term *pole* also in the general case for the point of which the y's are the coordinates.

Between the curves $v=0$ will be an infinite number having a double point, but only a finite number with two double points or a cusp. The condition for a double point of v is

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0,$$

where $v_i = \frac{1}{r}\frac{\delta v}{\delta x_i}$. By eliminating the x's between these equations, we obtain the discriminant Δ of v, of the order $3m(r-1)^2$ in the y's. Hence *the poles* (y) *of curves* v *with double points form a curve* $\Delta=0$ *of the order* $3m(r-1)^2$.

If we eliminate the y's between the above equations, we obtain *a function* K *of the* x*'s, the locus of the double points of the curves* v. *It is of the order* $3m^2(r-1)$.

The two curves Δ, K correspond point for point, to each point (y) on Δ corresponds one point (x) on K, the double point of the curve v for the pole (y). I shall prove:—*The poles* (y) *for which* v *has two double points are double points of* Δ; *and the poles* (y) *for which* v *has a cusp are cusps of* Δ.

Δ is the discriminant of v; hence we have, for any point (y) on $\Delta=0$, and for the corresponding point (x) on K,

$$\frac{\delta v}{\delta a_{\kappa\lambda\mu}}\frac{\delta \Delta}{\delta y_i} = \Delta_{\kappa\lambda\mu}\frac{\delta v}{\delta y_i}, \quad \text{where} \quad \Delta_{\kappa\lambda\mu} = \frac{\delta \Delta}{\delta a_{\kappa\lambda\mu}}.$$

* Salmon, Geometry of Three Dimensions, 2nd edit., p. 179, foot note; and Lessons introductory to the Modern Higher Algebra, 2nd edit, p. 149.

This shows at once that $\frac{\delta\Delta}{\delta y_i} = 0$, as soon as every $\Delta_{\kappa\lambda\mu} = 0$, or that Δ has a double point as soon as v has two double points; because the condition for two double points on v is, that all differentials of Δ with respect to the coefficients $a_{\kappa\lambda\mu}$ vanish, or that every $\Delta_{\kappa\lambda\mu} = 0$.

When every $\Delta_{\kappa\lambda\mu}$ vanishes, we have for the second differential coefficients $\frac{\delta^2\Delta}{\delta y_i \delta y_k}$ the formula

$$\left(\frac{\delta v'}{\delta a_{\kappa\lambda\mu}} \frac{\delta v''}{\delta a_{\kappa'\lambda'\mu'}}\right) \frac{\delta^2\Delta}{\delta y_i \delta y_k} = \Delta_{\kappa\lambda\mu,\,\kappa'\lambda'\mu'} \left(\frac{\delta v'}{\delta y_i}\frac{\delta v''}{\delta y_k} + \frac{\delta v''}{\delta y_i}\frac{\delta v'}{\delta y_k}\right),$$

where v' means v for the coordinates of the one, and v'' for the coordinates of the other double point. In case that these two double points coincide, v has a cusp. Then $v' = v''$, and our formula changes into

$$\frac{\delta v}{\delta a_{\kappa\lambda\mu}} \frac{\delta v}{\delta a_{\kappa'\lambda'\mu'}} \frac{\delta^2\Delta}{\delta y_i \delta y_k} = \Delta_{\kappa\lambda\mu,\,\kappa'\lambda'\mu'} \frac{\delta v}{\delta y_i}\frac{\delta v}{\delta y_k}.$$

This enables us to form the Hessian of Δ for the point (y), for which v has a cusp. We find it proportional to

$$\begin{vmatrix} \frac{\delta v}{\delta y_1}\frac{\delta v}{\delta y_1}, & \frac{\delta v}{\delta y_1}\frac{\delta v}{\delta y_2}, & \frac{\delta v}{\delta y_1}\frac{\delta v}{\delta y_3} \\ \frac{\delta v}{\delta y_2}\frac{\delta v}{\delta y_1}, & \frac{\delta v}{\delta y_2}\frac{\delta v}{\delta y_2}, & \frac{\delta v}{\delta y_2}\frac{\delta v}{\delta y_3} \\ \frac{\delta v}{\delta y_3}\frac{\delta v}{\delta y_1}, & \frac{\delta v}{\delta y_3}\frac{\delta v}{\delta y_2}, & \frac{\delta v}{\delta y_3}\frac{\delta v}{\delta y_3} \end{vmatrix}.$$

It vanishes identically; and so do all first minors. Hence Δ has a cusp. Q. E. D.

These properties of Δ are sufficient to determine its class. The double points and cusps of Δ are the solutions common to every $\Delta_{\kappa\lambda\mu} = 0$. Δ vanishes for the same values. Now, if $\Delta = 0$, the coordinates of the double point of v are determined by proportions, of which one is

$$x_1^r : x_1^{r-1}x_2 : \ldots\ldots : x_2^r = \Delta_{r,0,0} : \Delta_{r-1,1,0} : \ldots\ldots : \Delta_{0,r,0}.$$

These show, if $\Delta_{r,0,0} = 0$, either every $\Delta_{\kappa\lambda\mu}$ vanishes, or $x_1 = 0$. In the last case, $\Delta_{r,0,0}$ vanishes like x_1^r, or the curve $\Delta_{r,0,0}$ has here a multiple point with r branches. We see *the common solutions of every* $\Delta_{\kappa\lambda\mu}$ *are those solutions of* $\Delta = 0$, $\Delta_{r,0,0} = 0$ *for which* x_1 *is not* $= 0$. The double points (x) on v, for which $x_1 = 0$, are the points where the line $x_1 = 0$ meets the curve K; their number is $3m^2(r-1)$. As $\Delta_{r,0,0}$ has here r branches, the number of solutions common to Δ, $\Delta_{r,0,0}$, where

$x_1 = 0$ is $3m^2r(r-1)$. Subtracting this from the number of solutions of Δ, $\Delta_{r,0,0}$, or from $3m^2(r-1)^2(3r^2-6r+2)$, we obtain

$$3m^2(r-1)(r-2)(3r^2-3r+1)$$

as the number of solutions common to every $\Delta_{\kappa\lambda\mu} = 0$. Δ has here either a double point, in which case the curve $\Delta_{r,0,0}$ meets Δ in two points. Or Δ has a cusp, and then the curve $\Delta_{r,0,0}$ meets Δ in three points; because all curves $\Delta_{\kappa\lambda\mu}$ touch the tangent at a cusp of Δ. If we therefore denote the number of double points by δ and the number of cusps by κ, we have

$$2\delta+3\kappa = 3m^2(r-1)(r-2)(3r^2-3r+1).$$

It follows that *the curve* Δ *is of the class*

$$3m(r-1)(2mr-m-r+1).$$

Taking the curves v as polars to a curve u of the order $n = m+r$, we obtain curves K and Δ for the polars of each order, except the linear ones. These curves are related in the following way:—The curve Δ for the m^{th} polars is the curve K for the $(n-m-1)^{\text{th}}$ polars, and *vice versâ*. Steiner calls these curves "Conjugirte Kerncurven." For the first polars, or for $m = 1$, $r = n-1$, the curve K is the Hessian, and Δ is that curve which Cremona calls the Steinerian. I shall denote all these curves by S, with an index indicating for which polars they form the curve Δ. Then $S^{(m)}$ and $S^{(n-m-1)}$ are conjugate, and $S^{(n-2)}$ especially is the Hessian.

The curve $S^{(m)}$ *is the locus of the poles of* m^{th} *polars which have double points*, and *it is the locus of the double points of the* $(n-m-1)^{th}$ *polars. The conjugate curve* $S^{(n-m-1)}$ *is the locus of the double points of the* m^{th} *polars, and the locus of the poles of the* $(n-m-1)^{th}$ *polars with double points.* Their order and class follow directly from the numbers given before for Δ. These numbers are, for $m = 1$, given by Steiner without a proof. Cremona has determined them by geometrical, and Professor Clebsch by analytical methods. The other curves for m different from 1 are only mentioned by Steiner.* Cremona and Clebsch do not treat of them. We have further *the poles of* m^{th} *polars with two double points are double points of* $S^{(m)}$, *and the poles of the* m^{th} *polars with a cusp are cusps of* $S^{(m)}$. This theorem is also known for $m = 1$. Steiner gives it for the double points. Clebsch has guessed the theorem for cusps, and

* Steiner, Crelle's Journ., t. 47, p. 4. Cremona, Introd. ad una teoria geom. delle curve piane, Art. XX., 118 (*b*). Clebsch, Crelle's Journ., t. 64, Ueber einige von Steiner behandelte Curven.

Steiner, speaking of the curves $S^{(1)}$ and $S^{(n-2)}$, says:—"Diese Curven werden 'nebst andern' conjugirte Kerncurven genannt." These "other" curves are, no doubt, the curves $S^{(m)}$ and $S^{(n-m-1)}$.

proved it for curves of the fourth order. Cremona proves it for curves of any order.*

I add a theorem about the tangents of these curves.

The tangents of the curve $S^{(m)}$ *are the linear polars of the points on the conjugate curve* $S^{(n-m-1)}$ *with respect to the* $(m-1)^{th}$ *polars of the corresponding points on the curve* $S^{(m)}$, *and these points are the points of contact.* They are further *the linear polars of the points of contact with respect to the* $(n-m)^{th}$ *polars of the corresponding points on* $S^{(n-m-1)}$.

Cremona gives this also for $m=1$.† I shall prove it for $m=2$, the proof being the same in other cases.

Let $P_2 = y_1^2 u_{11} + y_2^2 u_{22} + y_3^2 u_{33} + 2y_2 y_3 u_{23} + 2y_3 y_1 u_{31} + 2y_1 y_2 u_{12}$ be the second polar of the point (y) with respect to the original curve u. $S^{(2)}$ is the discriminant of P_2, taking the x's as variables. Now for $S^{(2)}=0$ and the corresponding (x) on the conjugate curve $S^{(n-3)}$, we have

$$\frac{\delta P_2}{\delta a}\frac{\delta S^{(2)}}{\delta y_1} = \frac{\delta S^{(2)}}{\delta a}\frac{\delta P_2}{\delta y_1},$$

and

$$\frac{\delta P_2}{\delta y_1} = \frac{\delta P_1}{\delta x_1} = \frac{\delta P_{n-2}}{\delta y_1},$$

where P_1 denotes the first polar of the point (y), and P_{n-2} the conic polar of (x). The tangent to $S^{(2)}$ at the point (y) is therefore

$$z_1\frac{\delta P_2}{\delta y_1} + z_2\frac{\delta P_2}{\delta y_2} + z_3\frac{\delta P_3}{\delta y_3} = 0.$$

Instead of this we may write

$$z_1\frac{\delta P_1}{\delta x_1} + z_2\frac{\delta P_1}{\delta x_2} + z_3\frac{\delta P_1}{\delta x_3} = 0,$$

or

$$z_1\frac{\delta P_{n-2}}{\delta y_1} + z_2\frac{\delta P_{n-2}}{\delta y_2} + z_3\frac{\delta P_{n-2}}{\delta y_3} = 0.$$

The first of these equations is the linear polar of (x) with respect to P_1, the first polar of the point (y). The second equation represents the linear polar of (y) with respect to P_{n-2}, the conic polar of the point (x). This is the theorem.

$S^{(2)}$ is the locus of the double points (y) of the cubic polars; the conjugate curve $S^{(n-3)}$ is the locus of the corresponding poles (x). The conic polar of (x) on $S^{(n-3)}$ passes through the double point (y) of the cubic polar of (x), and the linear polar of the point (y) to this conic polar is its tangent. But it is also tangent to $S^{(2)}$. Hence the conic polars of the points on $S^{(n-3)}$ touch the curve $S^{(2)}$. And generally:—

The $(n-m)^{th}$ *polars of the points on* $S^{(n-m-1)}$ *touch the conjugate curve*

* Steiner, l. c. Clebsch, Crelle's Journ., t. 59, §§ 2, 6. Cremona, l. c., Art. XX., 120, 121.

† L. c., Art. XX., 118 (*a*); XXI., 127 (*a*).

$S^{(m)}$ *at the corresponding points. The* $(m+1)^{th}$ *polars of the points on* $S^{(m)}$ *touch the conjugate curve* $S^{(n-m-1)}$.

This theorem also is given by Cremona for $m=1$ and $m=n-2$.*

[Erratum.—Page 110, line 15, *for* $= \Xi_k \frac{\delta \Xi}{\delta \lambda}$ *read* $\Xi_k \frac{\delta f(\eta)}{\delta \lambda}$.]

Nov. 26th, 1868.

Prof. SYLVESTER, V.P., in the Chair.

Messrs. R. Abbay, G. H. Darwin, B.A., G. O. Hanlon, and T. Bond Sprague, M.A., were elected Members.

Mr. W. K. Clifford read the following paper:—

On a Generalization of the Theory of Polars.

[The present Note establishes the idea of the polar curve of a curve of given class in respect of a curve of given order, the class being less than the order; and of the polar curve of a curve of given order in respect of a curve of given class, the order being less than the class. It also deals with a certain invariant of two curves, such that the order of one is equal to the class of the other; and with certain other invariants and contravariants arising out of the theory of polars. I desire to present these ideas by themselves to the Society, because they seem likely to be useful for other purposes than that to which I propose to apply them subsequently, viz., the extension of Grasmann's Geometric Analysis.]

1. Let B_n be a curve of the nth order, and c_m a curve of the mth class. Let the equations of the curves be

$$B_n \equiv (A, B, C, D \ldots\ldots)(x\ y\ z)^n$$

$$c_m \equiv (a, b, c, d \ldots\ldots)(\xi\ \eta\ \zeta)^m$$

in point and line coordinates respectively.

In c_m write $\frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}$ in place of ξ, η, ζ respectively, and operate on B_n with the symbol thus formed. I denote the result by merely writing c_m as an operator before B_n; thus

$$c_m B_n \equiv \left(a, b, c \ldots\ldots\right)\left(\frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}\right)^m . (A, B, C, \ldots\ldots)(x, y, z)^n ;$$

then we find

(i.) If m is less than n, $c_m B_n$ is a covariant, which I call the *polar curve* of c_m in respect of B_n. It is given in the point coordinates x, y, z, and is of order $n-m$.

* L. c., XX., 118 (*a*); XXI., 127 (*a*).

(ii.) If m is equal to n, $c_n B_n$ is an invariant of the two curves. When this invariant vanishes, I say that the curves are *harmonic* of each other.

(iii.) If m is greater than n, $c_m B_n = 0$ always.

2. It is important to show that these new meanings of the words *polar* and *harmonic* include the old meanings. Now the mth polar of a point (x, y, z) whose tangential equation is $x, \xi+y, \eta+z, \zeta=0$, say the point P, is $\left(x, \frac{\delta}{dx}+y, \frac{\delta}{dy}+z, \frac{\delta}{dz}\right)^m . B_n = 0$;

which will be denoted in our present notation by $p^m B_n = 0$. But this is, according to the new definition, the polar of m times the point p; the point p taken m times being of course a particular case of a curve of the mth class. Again, the two conics

$$(a, b, c, f, g, h)(\xi\ \eta\ \zeta)^2, \qquad (A, B, C, F, G, H)(x, y, z)^2$$

have been called *harmonic* when

$$Aa+Bb+Cc+2Ff+2Gg+2Hh = 0,$$

the invariant being obviously the result of turning the first into an operator and applying it to the second.

3. Returning to the curves c_m, B_n, we may convert B_n into an operator by writing in it $\frac{\delta}{d\xi}, \frac{\delta}{d\eta}, \frac{\delta}{d\zeta}$ in place of x, y, z respectively. The result of the operation on c_m may be denoted by $B_n c_m$. As before, if $n>m$, the result is zero; if $n=m$, it is the harmonic invariant; if $n<m$, it is a curve of class $m-n$, which may be called the polar curve of B_n in respect of c_m.

4. Another definition of polar curves may be given. Suppose that the curves g_m, h_n together make up a curve which is harmonic of B_{m+n}. It is convenient to say that the curves g_m, h_n are *complementary*. Then it is clear that *all the curves complementary to a given curve are harmonic of its polar*. This may be regarded as a generalization of the theorem: All the points conjugate to a fixed point in regard to a conic, lie on the polar of the fixed point.

5. Consider a curve of even order B_{2n}. There is an invariant of the order $\frac{1}{2}(n+1)(n+2)$ in the coefficients, which vanishes when the nth differentials are in involution; this invariant is a symmetrical determinant. Its evectant is a contravariant of the order $2n$ in the variables, and $\frac{1}{2}(n^2+3n)$ in the coefficients, which may be called b_{2n}. The curves B_{2n}, b_{2n} are so related that *if* X_n *be the polar of* y_n *in respect of* B_{2n}, *then the* y_n *is the polar of* X_n *in respect of* b_{2n}. That is to say, if $X_n = y_n B_{2n}$, then $X_n b_{2n} = I \,.\, y_n$ where I is the invariant just defined. For example, if we consider a series of conics c_2 and their polars C_2 in regard to a given quartic curve Q_4, then there exists a curve q_4 of the fourth class, such that in respect to it the conics c_2 are the polars of C_2.

6. Two curves whose order and class are different may be made susceptible of the harmonic relation by taking each a proper number of times. Thus, curves B_6, c_8 have an invariant $(B_6)^4\,(c_8)^3$, of the order 4 in the coefficients of B_6 and of the order 3 in the coefficients of c_8.* It is to be observed that the equation $B_{mn}(c_m)^n = 0$, is the most general relation of the nth order that can subsist among the coefficients of c_m.

7. The following remarks relate to theorems in Dr. Henrici's paper "On certain formulæ concerning the Theory of Discriminants."

If the polar of C_m in respect of B_n has a node, C_m is harmonic of a curve of order $\zeta m\,(n-m-1)^2$, which is Dr. Henrici's curve $S^{(m)}$. In fact when C_m is a point taken m times, the point is *on* the curve $S^{(m)}$; this is Dr. Henrici's theorem.

In general, $x^m y^n B_{m+n+1}$ denotes a straight line. If it vanishes identically, x is a node on the nth polar of y, and y is a node on the mth polar of x. In this case $x^m y^{n-1} B_{m+n-1}$ and $x^{m-1} y^n B_{m+n-1}$ are conics having nodes at y and x respectively.† In the relation

$$x^m y^n B_{m+n+1} \equiv 0$$

write $x+\delta x$ for x and $y+\delta y$ for y. This is equivalent to supposing δx and δy to be points on the tangents at x and y to the curves $S^{(m)}$ $S^{(n)}$ which are the loci of those points respectively. Then we have

$$(my\delta x + nx\delta y)\, x^{m-1} y^{n-1} B_{m+n+1} \equiv 0\,;$$

operate on this with y; we know that

$$n\delta y\,.\,x^m y^n B_{m+n+1} = 0,$$

it follows that

$$m\delta x\,.\,x^{m-1} y^{n+1} B_{m+n+1} = 0\,;$$

that is to say, the tangent at x to $S^{(m)}$ is the line-polar of y in respect of the $(m-1)$th polar of x. This is another of Dr. Henrici's theorems. I have added this proof as an example of the readiness with which the operative notation lends itself to such investigations.

The Chairman next gave an account of "Spirals whose arcs and the arcs of whose Pedals are algebraically related."

The subject of this paper was connected with the author's papers in

* Thus we may have an invariant which vanishes when a curve is harmonic to itself. Let U_m be the curve, u_n its reciprocal: then $U_m^n u_n^m$ is the invariant in question, or a power of it. For a conic, it is the discriminant; for a cubic, the invariant T of the sixth order. See Salmon's "Higher Algebra," 1st ed., note to p. 67.

If the curve U_m has no node, the class $= m\,(m-1)$, and the invariant is $(U_m)^{m-1} u_{m(m-1)}$. If however U_m has a node d, d^2 is part of the reciprocal, and the invariant is $d^2 u_{m(m-1)-2}\,(U_m)^n$. Cusps may be similarly dealt with.

† Viz., these are the pairs of tangents at the two nodes. It is observable that the tangents at x to the nth polar of y, the tangent to the curve $S^{(m)}$, and the line xy, form a harmonic pencil.

the "Philosophical Magazine" for October and December, 1868, and was illustrated by Diagrams, which are given in the above-named December Part.

December 10th, 1868.

Prof. CAYLEY, President, in the Chair.

Mr. F. E. Ramsay, B.A., was elected a Member.

Mr. Archibald Smith, F.R.S., read a paper on "Graphic representations of the amount and direction of the forces which act on the Magnetic needle." These representations are designated by the general name of "Dygogram," a contraction for "Dynamo-goniogram," or Force and Angle Diagram.

(The substance of the Paper, with the illustrative diagrams, has been published in the third edition of the Admiralty Manual on the Deviations of the Compass).

Mr. Thomas Cotterill then read the following paper:—

On a Correspondence of Points, such that a curve of the n*th order in one plane corresponds to a curve of the* 4n*th in another plane, with three multiple points of the order* n *on the line of intersection of the planes, and three other multiple points of the order* 2n.

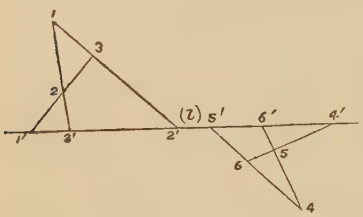
Preliminary Theorems.

In a plane, take three fixed points and the lines through them forming a triangle, and a fixed line (l) which does not pass through any of the points.

Then (A), the conics through the three points and a fourth point, cut the line (l) in pairs of points in involution, some or all of which are real. If the fourth point is taken arbitrarily, the double points of the involution on the line (l) (real or imaginary) are distinct. But if the point is taken on (l), the double points of involution coincide at this point, which is also a point of each pair of the involution determined on the line. So, if the point is taken on a line of the triangle, the double points on the line (l) coincide at the intersection of the lines, which is again a point of each pair of the involution.

Conversely (B), if the three points and line are taken as before, the system of conics through these three points and the several pairs of points of an involution on the line (l) will in general intersect again in only one other point, which is thus determined by a pair of conics. If the double points of the involution on the line (l) do not coincide, the fourth point will be some one point of the plane not lying on the

lines. Thus it may be a point consecutive to one of the fixed points in a given direction. If the double points coincide on (l) at a point which is not one of its intersections with the lines of the triangle, this point itself is the fourth point. But if the double points on (l) coincide at one of its intersections with a line of the triangle, the fourth point may be any point on this line.



1. Let 1 2 3, 4 5 6 be two triads of points in space, such that no three are in the same line, and no four in the same plane. Let the planes through the triads of points intersect in the line (l), which is cut by the lines 23, 31, 12, 56, 64, 45 respectively in the points $1', 2', 3', 4', 5', 6'$.

In the plane of the triangle 1 2 3, called the first plane, take any other point 7 except the points $4', 5', 6'$, and let two conics through 1 2 3 7 cut (l) in α, β and α', β' respectively. Then in the second plane, the plane of the triangle 456, the two conics $456\alpha\beta$ and $456\alpha'\beta'$ will again intersect in one unique point 8, which by (B) is the common remaining intersection of the conics through 4 5 6 and the several pairs of points of the involution on (l) determined by the two pairs $\alpha\beta$ and $\alpha'\beta'$. Thus by this construction a point in one plane in general determines one, and but one, point in the other plane, the property being such, that if 7 in the first plane gives 8 in the second, the point 8 in the second plane will give 7 in the first. Two such points are called corresponding points. If the point 7, considered as in the first plane, be any point on the line (l) except the points $4', 5', 6'$, the point corresponding to it in the second plane coincides with it, so that the line of intersection of the two planes is the locus of coincident corresponding points, and it is evident that corresponding points cannot coincide except on this line.

Let a point be called a critical point, if the construction fails to give one unique point corresponding to it. Then 1, 2, 3 and 4, 5, 6 are evidently critical points of their respective planes. Now let 7, considered as in the first plane, be one of the points $4', 5', 6'$; $4'$ for instance. Then the conics in the second plane which determine the point corresponding to $4'$ break up into two lines, one of which is always the line $4'56$ of the triangle 456 passing through it, and the other is any line in the plane through the point 4. Thus to the point $4'$ in the first plane, any point of the line $4'56$ in the second plane may correspond, and conversely the points of the line $4'56$ (except the critical points 5 and 6) have only one corresponding point in the first plane, the point $4'$. The points $4', 5', 6'$ are therefore critical points of the plane 123, to which the lines 56, 64, 45 respectively correspond in

the second plane, and all the points (except the critical points) on each of these lines have a common corresponding point, the intersection of the line on which the point lies with the line (l). So $1', 2', 3'$ are critical points of the plane 456 to which the lines 23, 31, 12 respectively correspond, and the points (except the critical points) on each of these lines have respectively the corresponding points $1', 2', 3'$.

2. Having thus examined the nature of the correspondence of the points (except 1 2 3 4 5 6) of the two planes, we can find the order of a curve in one plane, the points of which correspond to a given curve in the other plane. For a curve of the nth order in the plane 123 which does not pass through a critical point will cut the line (l) in n points which are also single points of the corresponding curve. Each side of the triangle 123 will also be cut by the curve n times, and to each of these n points, the intersection of the side with the line (l) corresponds, so that n branches of the corresponding curve will pass through each of the points $1', 2', 3'$. The line (l) thus contains n single points and three multiple points of the order n and no other point of the corresponding curve. The complete curve therefore to a curve of the nth order in the plane 123 is a curve of the $4n$th order with three multiple points of the order n at the points $1', 2', 3'$ of the plane 456. But the points 4, 5, 6 are also critical points of the same plane, and their peculiarities are found by observing that if α, β are points on (l), the non-critical points of the conics $123\alpha\beta$, $456\alpha\beta$ correspond, or the conics themselves correspond, but not completely, as a curve of the 8th order corresponds to a conic. Thus to the conic $1234'5'$ correspond the right lines $654', 645'$. But the points $4', 5'$ have been shown to correspond respectively to the non-critical points of these lines. If then 7 is a non-critical point on the conic $1234'5'$, to the system of conics through 1 2 3 7, conics through 4 5 6 touching the same line 6α at 6 must correspond, α being the second intersection of the conic $12376'$ with the line (l). Thus the points 4, 5, 6, are critical points of their plane, to which the conics $123\{5'6', 6'4', 4'5'\}$ respectively correspond, and the non-critical points of the conics correspond to points consecutively coincident with the points 4, 5, 6. We have already seen that $1', 2', 3'$ are critical points of the same plane, and that the lines 23, 31, 12 correspond to them. If the point 7 is a non-critical point of the line 12, and the line 37 meet (l) in α, then the point consecutive to $3'$ on the tangent to the conic $4563'\alpha$ at $3'$ corresponds to 7. Conversely, for a branch of a curve through a critical point, the corresponding curve breaks up into the right line or conic corresponding to the critical point, and the right line or conic is cut by the remaining portion of the corresponding curve in the point which corresponds to the consecutive point on the branch.

3. Having thus examined the peculiarities of the critical points $(1, 2, 3, 4', 5', 6')$ of the first plane, and $(4, 5, 6, 1', 2', 3')$ of the second

plane, by remembering that every other point of either plane has but one corresponding point in the other, and that to a curve of the order n (0_n) in the plane 123 corresponds a curve of the order $4n(0_{4n})$ in the plane 456 with multiple points of the order n at its critical points 1′, 2′, 3′, we can now add that the curve 0_{4n} has also multiple points of the order $2n$ at the critical points 4, 5, 6, the $2n$ branches at each of these points corresponding to the $2n$ points in which the conic corresponding to the critical point cuts the curve 0_n. Let 0_n $(1_a 2_b 3_c 4'_{a'} 5'_{b'} 6'_{c'})$ represent a curve of the nth order passing through a critical point (as 1) the number of times (a) denoted by the suffix of the critical point. Then to the curve 0_n in the plane 123 corresponds a curve 0_{4n} $(4_{2n} 5_{2n} 6_{2n} 1'_n 2'_n 3'_n)$ in the plane 456.

4. Let $\phi(n) = \frac{n(n+3)}{2}$, the number of points necessary to determine a curve of the nth order; $\mathrm{M}(k) = \frac{k(k+1)}{2}$, the number of points equivalent to a given point being a multiple point of the order k. Then $\phi(-n)+1 = \frac{n(n-3)}{2}+1 = \frac{(n-1)(n-2)}{2}$ is the greatest number of double points that a curve of the nth order can possess. Also $\mathrm{M}(-k) = \frac{k(k-1)}{2}$, the number of double points to which the multiple point is equivalent. Then

$$\phi(2n)-\phi(n) = \frac{2n(2n+3)}{2} - \frac{n(n+3)}{2} = 3\frac{n(n+1)}{2} = 3\mathrm{M}(n).$$

So
$$\phi(4n)-\phi(2n) = 3\mathrm{M}(2n),$$

and
$$\phi(4n)-\phi(n) = 3\{\mathrm{M}(n)+\mathrm{M}(2n)\},$$
$$\{\phi(-4n)+1\}-\{\phi(-n)+1\} = 3\{\mathrm{M}(-n)+\mathrm{M}(-2n)\},$$
$$\phi(4)-2 = 3\{\mathrm{M}(1)+\mathrm{M}(2)\}.$$

The meaning of these three last equations is obvious. The first shows that the number of points which determine a curve 0_n is the same as the number of points necessary to determine a curve 0_{4n} with three multiple points of the order n, and three other multiple points of the order $2n$, at given points. From the second equation we see that the deficiency of the two curves is the same, and from the last that curves of the 4th order passing through three single points and three double points form a réseau.

It may be remarked that in the same way we have

$$\phi(2^k n)-\phi(n) = 3\{\mathrm{M}(n)+\mathrm{M}(2n)+\ldots+\mathrm{M}(2^{k-1}n)\},$$

an equation admitting of transformations similar to those already given. Some very interesting properties of this class of equations are given by Cremona in the Memoirs of the Academy of Bologna for the years 1863 and 1865.

5. Lastly, let $0_n \{1_a 2_b 3_c 4'_{a'} 5'_{b'} 6'_{c'}\}$ a curve in the first plane, and $0_{\nu} \{4_\alpha 5_\beta 6_\gamma 1'_{\alpha'} 2'_{\beta'} 3'_{\gamma'}\}$ a curve in the second plane, be two curves in which the non-critical points correspond. The complete curve corresponding to the 0_n is always a curve $0_{4n}\ \{4_{2n} 5_{2n} 6_{2n} 1'_n 2'_n 3'_n\}$ which breaks up into the proper curve 0_ν, together with conics and right lines corresponding to the critical points 1, 2, 3, 4′, 5′, 6′, viz. the conics $(4562'3')_a$, $(4563'1')_b$, and $(4561'2')_c$, and the right lines $(56)_{a'}$, $(64)_{b'}$, and $(45)_{c'}$; where the suffix denotes the number of times the conic or right line is taken. For the remaining curve 0_ν, we must therefore have $\nu = 4n - 2\Sigma a - \Sigma a'$;

$$\alpha = 2n - \Sigma a - b' - c'; \quad \alpha' = n - b - c,$$
$$\beta = 2n - \Sigma a - c' - a'; \quad \beta' = n - c - a,$$
$$\gamma = 2n - \Sigma a - a' - b'; \quad \gamma' = n - a - b.$$

But, as the curves correspond, the letters in italics and the corresponding Greek letters may be interchanged. Thus it will be found that

$$n = 4\nu - 2\Sigma a - \Sigma a'; \ \&c.$$

6. If, as before, 7 is a point in the first plane corresponding to the point 8 in the second plane, we have seen that the points 1, 2, 3, 7 and 4, 5, 6, 8 are such that conics can be drawn through them cutting twice on the line (l). Such pairs of conics are twisted quartics of the first species (the intersection of two quadric surfaces) having two actual double points. Hence the eight points (1 2 ... 8) form the basis of a faisceau of twisted quartics of the first species, so that the quartic is determined if it passes through any other point. If the ninth point is taken on the line joining two corresponding points 7, 8, three points of the quartic are collinear, so that the quartic degenerates into this line and the twisted cubic through the points 1, 2, 3, 4, 5, 6. And I may add that the correspondence given in the present paper arose from a consideration of a more general correspondence connected with the twisted cubic which passes through six points in space.

As it is always desirable to view a subject under different aspects, the author is happy in being permitted to append the following observations by Professor Cayley and Mr. W. K. Clifford on the connexion of the transformation with Cremona's general theory, and the analytical formulæ.

According to Cremona's general theory, — taking (x_1, y_1, z_1) and (x_2, y_2, z_2) as current coordinates in the two planes respectively,—if we take in the first plane, any three points 1, 2, 3, and any other three points 4′, 5′, 6′, then if $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$ are quartic curves, each having the double points 1, 2, 3, and the simple points 4′, 5′, 6′, we have a transformation $x_2 : y_2 : z_2 = X_1 : Y_1 : Z_1$

leading to a converse system

$$x_1 : y_1 : z_1 = X_2 ; Y_2 : Z_2$$

of the like form; viz., there will be in the second plane three points 4, 5, 6, and three other points 1′, 2′, 3′, such that $X_2 = 0$, $Y_2 = 0$, $Z_2 = 0$, are quartics having the double points 4, 5, 6, and the simple points 1′, 2′, 3′.

Analytically, Cremona's transformation is obtained by assuming the reciprocals of x_2, y_2, z_2 to be proportional to linear functions of the reciprocals of x_1, y_1, z_1—(of course, this being so, the reciprocals of x_1, y_1, z_1 will be proportional to linear functions of the reciprocals of x_2, y_2, z_2). Solving this under the theory as above explained, write

$$\left.\begin{matrix} \frac{1}{x_2} \\ : \frac{1}{y_2} \\ : \frac{1}{z_2} \end{matrix}\right\} = \left\{\begin{matrix} \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} \\ : \frac{d}{x_1} + \frac{e}{y_1} + \frac{f}{z_1} \\ : \frac{g}{x_1} + \frac{h}{y_1} + \frac{i}{z_1} \end{matrix}\right\} = \left\{\begin{matrix} P_1 \\ : Q_1 \\ : R_1 \end{matrix}\right.$$

if

$$\begin{aligned} P_1 &= ay_1z_1 + bz_1x_1 + cx_1y_1, \\ Q_1 &= dy_1z_1 + ez_1x_1 + fx_1y_1, \\ R_1 &= gy_1z_1 + hz_1x_1 + ix_1y_1. \end{aligned}$$

Hence

$$x_2 : y_2 : z_2 = Q_1R_1 : R_1P_1 : P_1Q_1.$$

$Q_1R_1 = 0$, &c., are quartics, or generally $\alpha Q_1R_1 + \beta R_1P_1 + \gamma P_1Q_1 = 0$ is a quartic, having three double points $(y_1 = 0,\ z_1 = 0)$, $(z_1 = 0,\ x_1 = 0)$, $(x_1 = 0,\ y_1 = 0)$, and having besides the three points which are the remaining points of intersection of the conics $(Q_1 = 0,\ R_1 = 0)$, $(R_1 = 0,\ P_1 = 0)$, $(P_1 = 0,\ Q_1 = 0)$ respectively; viz., these last are the points

$$\frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1} = ei - hf : fg - id : dh - ge, \quad \&c.\ \&c.$$

The double and simple points are fixed points (that is, independent of α, β, γ), and the formulæ come under Cremona's theory. It is, however, necessary to show that if the points 4′, 5′, 6′ are in a line, the points 1′, 2′, 3′ are also in a line. This may be done as follows:

Let there be three planes A, B, C, and let the points of the first two correspond by ordinary triangular inversion in respect of the triangle α_1 on the plane A, and β_1 on the plane B. Let also the planes B, C correspond by ordinary triangular inversion in respect of the triangle β_2 on the plane B, and γ_2 on the plane C. Then the correspondence between A and C is the one considered, the points 1 2 3 forming the triangle α_1 and the points 4 5 6 forming the triangle γ_2. The points

$4'\,5'\,6'$ and $1'\,2'\,3'$ in the planes A, C respectively correspond to the triangles β_1, β_2; and the conditions that $4', 5', 6'$ shall be in a line and that $1', 2', 3'$ shall be in a line, are the same condition, namely, that the triangles β_1, β_2 shall be inscribed in the same conic. Analogous properties must apparently belong to Cremona's other transformations, and the investigation of them will form an interesting part of the theory.

It is important, also, to notice the relation of the transformation to Hesse's "Uebertragungsprincip," Crelle, Vol. 66, p. 15, which establishes a correspondence between the points of a plane and the point-pairs on a line. If $Ax^2+2Bxy+Cy^2=0$ is the equation of a point-pair, the co-ordinates in the plane are taken by Hesse *directly*, but in the present Paper *inversely* proportional to A, B, C.

The President and Professor Hirst made some remarks on the Paper, the latter pointing out its connection with his method of Quadric Inversion.

The following presents were received:—

"Société Philomathique de Paris. Extrait de la séance du 18 Janvier, 1868;" from Prof. Hirst.

"Proceedings of the Literary and Philosophic Society, Oct. 1868. Recherches sur les surfaces réglées tétraédrales Symmetriques, par Jules de la Gournerie;" from Prof. Cayley.

"Crelle," Band 69, 2es Heft.

"On the curves which satisfy given conditions;" by Prof. Cayley, from the Author.

"A Photographic Diagram consisting of 10,206 lines connecting 144 points on a circle, and involving 17,178,876 intersections, to illustrate Mr. Woolhouse's Question in the Lady's and Gentleman's Diary for 1867;" from Mr. Woolhouse.

January 14th, 1869.

Prof. CAYLEY, in the Chair.

Mr. G. H. Darwin, B. A., was admitted into the Society.

Mr. S. Roberts then read a paper

On the Mechanical Description of some species of Circular Curves of the third and fourth degrees.

"At æquatio non est, sed descriptio, quæ curvam Geometricam efficit."—*Newton.*

In the present state of the theory of quartic and cubic curves, it is very desirable that we should be able to draw them continuously.

There are to be found in the books several isolated examples of such description. To cite a familiar and ancient instance, Nicomedes drew the conchoid, named after him, by means of a simple instrument called the "Trammel." This curve is generated, it will be remembered, by any point on a straight line which constantly passes through a fixed point, while another point upon it is constrained to move rectilinearly. The instrument, therefore, consists of a grooved bar moving over a fixed pin, and at the same time turning on another pin which travels along a straight groove. The method given by Newton, for drawing the Cissoid of Diocles, is similar. In this case, one leg of a right angle always passes through a fixed point, while a point on the other leg moves along a straight line. When the distance of the fixed point from the given line is equal to the length intercepted between the angle and the point constrained to move rectilinearly, the middle point of the intercepted length will generate the Cissoid. In effect, Newton generalizes this by considering the case of a point on the leg not being the middle point of the intercepted length. I may mention also the Limaçon of Pascal, which is generated by making a straight line pass constantly through a given point upon a circle. A point on the line at a constant distance from its other intersection with the circle will describe the Limaçon, that is to say, a Cartesian oval having a finite double point.

And it is to be observed, that in this instance the curve is general, except as to its essential singularities. For the double point is the fixed point, and, being given, is equivalent to three conditions. We have also the two constants of the circle and the arbitrary distance of the generating point. Now two given cusps and a given finite double point determine 11 of the 14 conditions which define a quartic curve, so that there remain only 3 disposable constants.

I have endeavoured to obtain results similar to the foregoing, having regard to mechanical convenience, and employing only points, straight lines, and circles. As might be anticipated, the curves which I am able to describe by these means are circular. The system of equations from which the equation representing the locus is evolved, is usually circular with respect to the uneliminated variables, and since the so called circular points at infinity satisfy all such equations, the result contains a circle, or the straight line at infinity, or the circular points without degeneration.

What follows, therefore, may be considered as an extension of the principle of the compasses to curves of the third and fourth degrees, and notwithstanding practical limitations the curves capable of being thus described are of considerable variety and elegance.

1. I find that the "Trammel" of Nicomedes may be generalised in three particulars without elevating the algebraical degree of the locus.

We may substitute for the fixed point or pole through which the moving line passes a fixed circle touched by the moving line.

We may substitute for the point which moves rectilinearly a circle touched by the moving line and having its centre always on the given line or "rule."

We may attach an arm to the moving tangent of given length and inclination.

Although I did not arrive at these conclusions *per saltum*, it is better here to take at once the most general case.

2. The centre of the fixed circle is taken as origin of rectangular coordinates; x_1, y_1 are the coordinates of the centre of the moving circle, x, y are the current coordinates of the locus, k is the distance of the generating point from the centre x_1y_1, and m is the tangent of the angle which the attached arm makes with the moving tangent.

We have therefore the following conditions in which X, Y are general coordinates.

The line $AX+BY+C=0$ is to touch the circles $X^2+Y^2=r^2$, $(X-x_1)^2+(Y-y_1)^2=s^2$.

Also we may write $A'(x-x_1)+B'(y-y_1)=0$,

$$m=\frac{BA'-AB'}{AA'+BB'},$$

$$(x-x_1)^2+(y-y_1)^2=k^2,$$

$$ax_1+by_1+c=0.$$

Putting L, M for $x-x_1$, $y-y_1$, respectively, we must eliminate A, B, C, A′, B′, L, M, from

$$(1)\quad r^2(A^2+B^2)=C^2,$$
$$(2)\quad s^2(A^2+B^2)=\{AL+BM-Ax-By-C\}^2,$$
$$(3)\quad L^2+M^2=k^2,$$
$$(4)\quad A'L+B'M=0,$$
$$(5)\quad aL+bM-ax-by-c=0,$$
$$(6)\quad A(mA'+B')+B(mB'-A')=0,$$

By (4) and (6), $A(mM-L)-B(mL+M)=0$;

and therefore, by (1) and (2),

$$s^2(1+m^2)(L^2+M^2)=\{m(L^2+M^2)-x(mL+M)-y(mM-L)-r\sqrt{(1+m^2)(L^2+M^2)}\}^2;$$

that is to say, by (3), $(mx-y)L+(my+x)M=t$,

where $t=mk^2+(s-r)k\sqrt{(1+m^2)}$.

By this and (5),
$$L=\frac{P(my+x)-bt}{a(my+x)-b(mx-y)},$$
$$M=-\frac{P(mx-y)-at}{a(my+x)-b(mx-y)},$$

where P is put for $ax+by+c$;

so that finally, by (3), the required locus is given by

$$\{P(my+x)-bt\}^2+\{P(mx-y)-at\}^2=k^2\{a(my+x)-b(mx-y)\}^2 \quad \ldots\ldots\ldots\ldots(A),$$

a circular curve of the fourth degree.

The equation may also be written in the form

$$(yQ-bt)^2+(xQ-at)^2=k^2(ay-bx)^2 \quad \ldots\ldots\ldots\ldots\ldots (B),$$

Q being a general linear function.

We may now suppose that $Q=0$ is a line parallel to the axis of x, and the equation is

$$\{y(y+c)-bt\}^2+\{x(y+c)-at\}^2=k^2(ay-bx)^2 \quad \ldots\ldots\ldots(C).$$

3. From equation (C) we derive, according to the common notation,

$$\frac{du}{dx}=2(y+c)\{x(y+c)-at\}+2k^2b(ay-bx),$$

$$\frac{du}{dy}=(4y+2c)\{y(y+c)-bt\}+2x\{x(y+c)-at\}-2k^2a(ay-bx).$$

These and the original equation are satisfied by the coordinates of the point where the line $y+c=0$ meets the line at infinity, and also by the coordinates of two points on the line $ay-bx=0$. For

$$y(y+c)-bt=0, \quad x(y+c)-at=0, \quad ay-bx=0 \quad \ldots\ldots\ldots(a)$$

have two common sets of roots.

The equation represents, therefore, a unicursal quartic curve, having a double point at infinity. The double focus and the two double points at a finite distance lie in a straight line. The equation is general for a quartic so conditioned; for, after determining the double point at infinity, we have at our disposal one constant of the "rule" line, three constants of the arbitrary circle, and the quantities k and m, making six in all. But the curve is subjected to eight known conditions, the linear relation of the double focus and the two finite double points being one.

If $b=0$ in (C), or $m=0$ in (A), there are two double points united at infinity, and the line $y+c=0$, or in (A) the line $P=0$, passes through four consecutive points. There is, however, a double point at a finite distance; and if $s-r=0$, the curve is the conchoid of Nicomedes, which consequently is not the most general curve of its kind. When $c=0$, $b=0$, $a=1$, the "rule" line passes through the centre of the given circle, and we have a peculiar special case.

The equation then is $(xy-t)^2+y^2(y^2-k^2)=0$.

4. In a practical point of view, it is convenient that the radii of the circles enter only by their sum or difference. We may consequently reduce either to a point without affecting the locus, or, what is more

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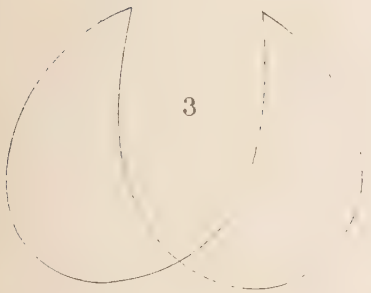

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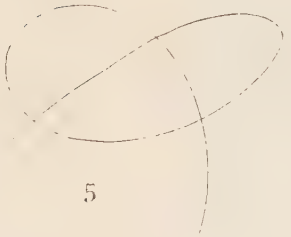

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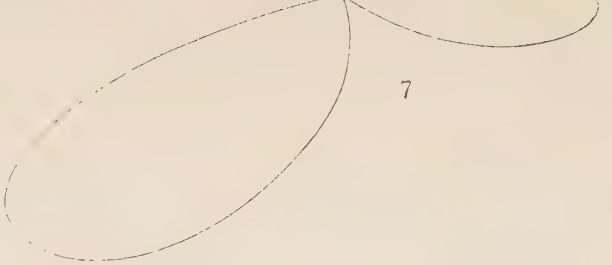

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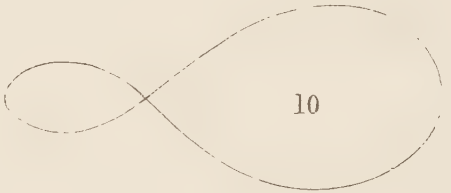

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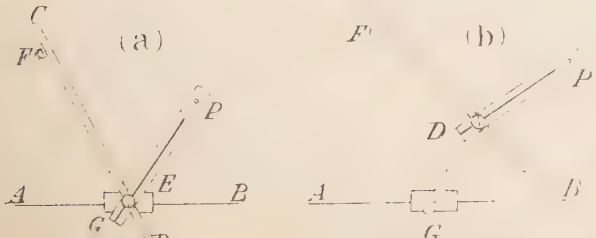

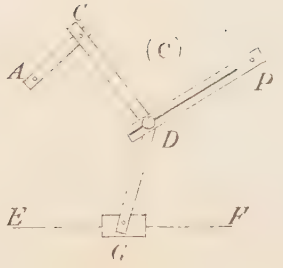

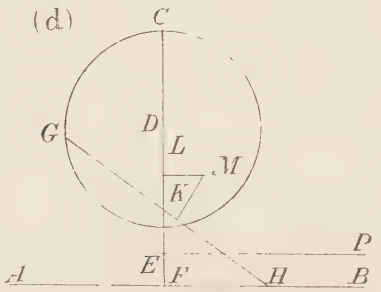

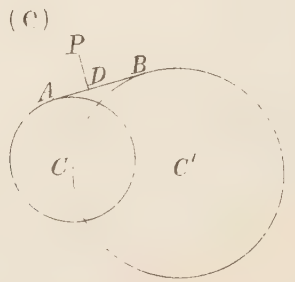

important, can make compensation for the sensible breadth of the moving bar.

The few figures which accompany this paper will give, though inadequately, some idea of the nature of the curves drawn in the way proposed. Fig. (1) represents a quartic curve having three real double points, two of them visible, the other at infinity. The common solutions of (*a*) may, however, be imaginary.

5. The quartic locus can be reduced to a cubic curve and a straight line. This straight line must manifestly be parallel to the given fixed line, since the distance of a point on the locus from that line cannot exceed k, a finite quantity. Referring to equation (A), and for simplicity's sake making $a=0$, $b=1$, we find the following conditions that a line parallel to the axis of x may be contained in the locus. They are

$$y^2+(y^2-k^2)\,m^2=0,$$
$$ty-mk^2\,(y+c)=0,$$
$$\{t-my\,(y+c)\}^2-(y^2-k^2)\,(y+c)^2=0,$$
$$t=mk^2+(s-r)\,k\,\sqrt{(1+m^2)}.$$

These equations are uniquely satisfied by $s-r=0$, $y=\dfrac{km}{\sqrt{(1+m^2)}}$. Thus, in order that the curve described may be a circular cubic, the necessary and sufficient condition is that the difference of the radii of the circles must be equal to the distance of the fixed line from the centre of the fixed circle. It will be seen that Newton's method of drawing the Cissoid fulfils this condition, which is also evident geometrically. The word "difference" may be taken in its algebraical sense.

Writing in (A) $\tan\theta$ for m, putting c for $s-r$, and still taking $a=0$, $b=1$, we have

$$\{(y+c)\,(y\sin\theta+x\cos\theta)-k\,(c-k\sin\theta)\}^2$$
$$+\{(y+c)^2-k^2\}\,(x\sin\theta-y\cos\theta)^2=0.$$

After some easy reductions, we can divide out $y+c+k\sin\theta$, leaving

$$(y+c-k\sin\theta)\,(x^2+y^2)-k^2\,(y-c-k\sin\theta)-2kc\,(y\sin\theta+x\cos\theta)=0 \quad \ldots\ldots\ldots\ldots \text{(D)},*$$

a general equation for circular cubics having a double point, that is to say, the "pedal curve" of a parabola.

Suppose, in fact, that we take a line parallel to the real asymptote for the axis of x, the equation of a circular cubic may be written

$$(y+\mathrm{A})\,(x^2+y^2)+\mathrm{B}x+\mathrm{C}y+\mathrm{D}=0.$$

* Besides the Cissoid and other known curves, Dr. Booth's "Logocyclic Curve" is included in the class here represented. We have to make $\theta=\frac{\pi}{2}$, $c=-k$.

Let x_1, y_1 be the coordinates of the double point, then we have

$$x_1^2+y_1^2+2y_1\,(y_1+\mathrm{A})+\mathrm{C}=0,$$
$$\mathrm{A}\,(x_1^2+y_1^2)+2\mathrm{B}x_1+2\mathrm{C}y_1+3\mathrm{D}=0,$$
$$2x_1\,(y_1+\mathrm{A})+\mathrm{B}=0;$$

or putting $c-k\sin\theta$ for A, we have

$$\mathrm{B}=-2kc\cos\theta,$$
$$\mathrm{C}=-2kc\sin\theta-k^2,$$
$$\mathrm{D}=\quad k^2\,(c+k\sin\theta).$$

These results agree with the equation (D).

Fig. (2) represents a circular cubic curve drawn in the manner above described.

7. An instrument fulfilling the conditions indicated can be easily made.

In Fig. (*a*), AB represents a rather stiff wire stretched across a board of convenient size. CD is a bar caused to touch a cylinder F by means of an elastic band. At E is a pin fixed in a short grooved bar G, while an arm EP carries a pencil at P, and is fixed to CD by a screw in any required position. It is also well to make CD capable of carrying a pencil.

In Fig. (*b*), GD represents a bar capable of moving about G, while G moves along the wire AB. The bar CD, which touches the cylinder F, and the arm PD, which carries a pencil at P, are clamped at D.

In a more elaborate instrument, the breaks which the above simple arrangements necessitate might be avoided by raising the moving apparatus above the level of the paper in the manner of Suardi's geometric pen.

8. If, instead of a fixed line, we take another fixed circle, the corresponding locus is usually of the sixth degree. When, however, the new fixed circle passes through the centre of the original fixed circle, and $s-r=0$, or the radii of the circles are equal, the locus contains a circle, as is geometrically evident. The remaining part of the locus is then a Limaçon. I have already remarked that the well-known mode of drawing the Limaçon is general. We only gain, therefore, in convenience by the extended mode of generation.

It is, perhaps, unnecessary to give here the details of the elimination, which are merely tedious. I have found the equation in its general form to be

$$\{(my-x+m\mathrm{B})^2+(mx+y+m\mathrm{A})^2\}\,(x^2+y^2-k^2)$$
$$+2\,\{m\,(\mathrm{B}x-\mathrm{A}y)-x^2-y^2\}\,\{\mathrm{A}\,(my+x)-\mathrm{B}\,(mx-y)+2x^2+2y^2\}$$
$$+\{\mathrm{A}\,(my+x)-\mathrm{B}\,(mx-y)+2x^2+2y^2\}^2=0;$$

the equation of the second fixed circle is

$$x^2+y^2+Ax+By=0.$$

It will be remembered that the Limaçon can also be generated as an epitrochoid, or as the "pedal curve" of a circle.

9. The next method which I propose is also adapted to mechanical application. I find that if one point of a line moves in a circle, while another point of the line moves in a straight line, any point rigidly connected with the moving line will describe a curve of the fourth degree.*

Let the coordinates of the points upon the circle and line be x_1, y_1, x_2, y_2 respectively, so that we may take

$$x_1^2+y_1^2=r^2,$$
$$ax_2+by_2+c=0.$$

Let l be the length of a perpendicular to the moving line, and let its extremity be considered as the generating point. If p, q are the parts into which the constant intercepted length is divided by the foot of the perpendicular l, and x', y' are the coordinates of the last-mentioned point, it is plain that

$$\frac{y_1-y'}{y'-y_2}=\frac{x'-x_1}{x_2-x'}=\frac{p}{q},$$

$$x=x'+\frac{l}{p+q}(y_1-y_2), \quad y=y'+\frac{l}{p+q}(x_1-x_2).$$

From which, writing L, M for $x-x_1$, $y-y_1$, L_2, M_2 for x_2-x_1, y_2-y_1, we get

$$(1) \qquad (p+q)\,L=pL_2-lM_2,$$
$$(2) \qquad (p+q)\,M=pM_2+lL_2.$$

We have also

$$(3) \qquad a(L_2-L+x)+b(M_2-M+y)+c=0,$$
$$(4) \qquad L^2+M^2=l^2+p^2,$$
$$(5) \qquad L^2+M^2-2xL-2yM+x^2+y^2-r^2=0.$$

By (1), (2), (3) we get

$$a\{(p+q)(pL+lM)-(l^2+p^2)(L-x)\}$$
$$+b\{(p+q)(pM-lL)-(l^2+p^2)(M-y)\}+(l^2+p^2)c=0;$$

which, since l, p, q are independent, may be written

$$\alpha L+\beta M+ax+by+c=0.$$

Also by (4) and (5),

$$2xL+2yM-(x^2+y^2-r^2+l^2+p^2)=0;$$

* If we take two straight lines instead of the circle and line, we get the "Elliptic Trammel." The locus degenerates to an ellipse.

and, eliminating L, M successively,

$$2(\alpha y+\beta x)\,\mathrm{L}+2y\mathrm{P}+\beta\phi=0,$$
$$2(\alpha y-\beta x)\,\mathrm{M}-2x\mathrm{P}-\alpha\phi=0,$$

where P is put for $ax+by+c$, and ϕ for $x^2+y^2-r^2+l^2+p^2$.

Finally, by (4),

$$(2y\mathrm{P}+\beta\phi)^2+(2x\mathrm{P}+\alpha\phi)^2=4(l^2+p^2)^2(\alpha y-\beta x)^2 \ldots\ldots\ldots(\mathrm{E}).$$

10. If we compare our last result with equation (B) of § 2, there is seen to be a remarkable analogy. And, in point of fact, the last-named equation is a particular case of the present one. Mechanically, however, it is not so. Some of the quantities which were finite are now infinite.

Let α, β be indefinitely small, while

$$\frac{\alpha}{2}(r^2-l^2-p^2)=a't, \qquad \frac{\beta}{2}(r^2-l^2-p^2)=b't,$$
$$(l^2+p^2)\,\alpha=a'k, \qquad (l^2+p^2)\,\beta=b'k,$$

then the present equation becomes identical in form with (B). Thus the two methods of description are connected in an unanticipated way.

The equation (E) represents a quartic curve having two double points on the line $\alpha y-\beta x=0$; for if we write the equation in the form

$$\psi^2+\chi^2-k^2(\alpha y-\beta x)^2=0,$$

then $\psi=0$, $\chi=0$, $\alpha y-\beta x=0$ have two common solutions.

We have at our disposal the two constants of P, the three constants of the circle α, β, and k, being eight in all. The quartic is therefore subjected to six conditions. It passes through the circular points and has two double points, and these and the double focus are connected by a linear relation. The remaining condition does not appear to be of a critical kind.

Practically we cannot translate the line $\alpha y-\beta x=0$ to infinity. The curve has no real points at infinity, except in limiting cases which practically cannot occur, though they may be approximated to.

11. The following easy construction enables us to draw a variety of circular quartics with two double points, or three double points, or a triple point:—

Three arms are fixed at D (Fig. *c*) by means of a screw. One arm carries a pencil P, another is attached to a radius AC which moves about A as centre, the third is attached to a short grooved bar G which moves with it along the line EF.

In this Paper, I have not thought it expedient to enter upon a detailed discussion of the equations which have been obtained.* I may remark, however, that we can determine a third double point by making the distance of the centre of the fixed circle from the fixed line equal to the length $p+q$ plus or minus the radius.

Thus, if CGD, AB (Fig. *d*) be the circle and line, CF being the position of the moving bar when it is perpendicular to AB, the extremity P of a perpendicular to CF at E will generate a curve with a double point at P. Moreover, it does not necessarily lie on a straight line through D, the centre of CGD, and another double point. Hence we shall have three double points. We may also arrange for a triple point. In fact, let GH be another position of the moving bar, which meets CF in K. Take $CL = GK$; then, if we raise a perpendicular at the middle point of LK, and another at the same distance from K on KH, they will meet in a point M, which will determine the required length of arm. The point M will be a triple point on the locus, for we have provided that another branch of the curve shall pass through it.

Figs. (3), (4), (5), (6), (7), (8) illustrate the curves drawn by this method. Fig. (6) is analogous to the curve which Plücker traced from its equation, and used for the purpose of showing that the twenty-eight double tangents of a quartic curve may be all real.

12. It remains to notice a mode of description which bears to the last a relation similar to that which the mode of describing the Limaçon bears to the generalized "Trammel" of Nicomedes.

Let a circle C′ (Fig. *e*) pass through the centre of the circle C, and let a line AB, equal in length to the radius of C, rest with its extremities on C and C′ respectively; I say that any point rigidly connected with AB will describe a quartic curve.

Taking the centre of C for the origin of rectangular coordinates, and the line joining the centres for the axis of x; writing also l for the perpendicular DP, p for AD, q for BD; x_1y_1 for the coordinates of A, x_2y_2 for the coordinates of B, L, M for $x-x_1$, $y-y_1$ respectively; L′, M′ for $x-x_2$, $y-y_2$ respectively; we have, in the same way as in § 9, the following equations:—

$$(1) \ldots\ldots (l^2+p^2)\,\mathrm{L}' + (pq-l^2)\,\mathrm{L} + rl\mathrm{M} = 0,$$

$$(2) \ldots\ldots (l^2+p^2)\,\mathrm{M}' + (pq-l^2)\,\mathrm{M} - rl\mathrm{L} = 0,$$

$$(3) \ldots\ldots (\mathrm{L}-x)^2 + (\mathrm{M}-y)^2 = (p+q)^2 = r^2,$$

$$(4) \ldots\ldots \mathrm{L}^2 + \mathrm{M}^2 = l^2 + p^2.$$

* It may be worth while to notice that the method of M. Chasles for drawing tangents to certain classes of curves is applicable to all the curves here dealt with.

Also, by means of

$$x_1^2+y_1^2=(x_1-x_2)^2+(y_1-y_2)^2=r^2,$$
$$x_2^2+y_2^2-2ax_2=0,$$

we have $$(x_1-a)\,x_2+y_1y_2=0,$$

or, what is the same thing,

$$(x-\mathrm{L}-a)\,(\mathrm{L}'-x)+(y-\mathrm{M})\,(\mathrm{M}'-y)=0;$$

and substituting the values of L′, M′ given by (1), (2), we get

$$\{(pq-l^2)\,(x-a)-(l^2+p^2)\,x-rly\}\,\mathrm{L}$$
$$+\{(pq-l^2)\,y-(l^2+p^2)\,y+rl\,(x-a)\}\,\mathrm{M}$$
$$+(l^2+p^2)\,\{x\,(x-a)+y^2-pq+l^2\}=0.$$

Also, by (3), (4), $2x\mathrm{L}+2y\mathrm{M}-x^2-y^2+r^2-l^2-p^2=0.$

By the successive elimination of L, M,

$$\{rl\,[x\,(x-a)+y^2]+a\,(pq-l^2)\,y\}\,2\mathrm{M}$$
$$+2\,(l^2+p^2)\,x\,[x\,(x-a)+y^2-pq+l^2]$$
$$+\{(pq-p^2-2l^2)\,x-rly-a\,(pq-l^2)\}\,(x^2+y^2-r^2+l^2+p^2)=0;$$

$$\{rl\,[x\,(x-a)+y^2]+a\,(pq-l^2)\,y\}\,2\mathrm{L}$$
$$-2\,(l^2+p^2)\,y\,[x\,(x-a)+y^2-pq+l^2]$$
$$-\{(pq-p^2-2l^2)\,y+rl\,(x-a)\}\,\{x^2+y^2-r^2+l^2+p^2\}=0.$$

The substitution of the values of L, M in (4) gives

$$\left\{\begin{matrix}[(pq-p^2-2l^2)\,y+rl\,(x-a)]\,[x^2+y^2-r^2+l^2+p^2]\\+2\,(l^2+p^2)\,y\,[x\,(x-a)+y^2-pq+l^2]\end{matrix}\right\}^2$$
$$+\left\{\begin{matrix}[(pq-p^2-2l^2)\,x-rly-a\,(pq-l^2)]\,[x^2+y^2-r^2+l^2+p^2]\\+2\,(l^2+p^2)\,x\,[x\,(x-a)+y^2-pq+l^2]\end{matrix}\right\}^2$$
$$=4\,\{l^2+p^2\}\,\{rl\,[x\,(x-a)+y^2]+a\,(pq-l^2)\,y\}^2.$$

Put χ for $x^2+y^2-l^2-q^2$, then the equation becomes

$$\{(pry+rlx-arl)\,\chi-2\,(l^2+p^2)\,axy+2r^2l^2y-2\,(pq-l^2)\,rlx+2\,(pq-l^2)\,arl\}^2$$
$$+\left\{\begin{matrix}[prx-rly-a\,(pq-l^2)]\,\chi-2\,(l^2+p^2)\,ax^2+2r^2l^2x\\+2\,(pq-l^2)\,rly+2a\,(pq-l^2)^2\end{matrix}\right\}^2$$
$$=4\,(l^2+p^2)\,\{rl\,[x\,(x-a)+y^2]+a\,(pq-l^2)\,y\}^2.$$

Now, if we consider successively the terms of this equation not involving χ explicitly, and multiplied by a^2, a^1, a^0, we find that each such term is in fact divisible by χ; and, by a somewhat tedious reduction, the following result will be obtained, after dividing out χ:

$$\{r^2(x^2+y^2)-2arly-2arqx+a^2\,(l^2+q^2)\}\,\{x^2+y^2-l^2-q^2-4pq+4l^2\}$$
$$-4\,(ax-rq)\,\{pr\,(x^2+y^2)-arly-a\,(pq-l^2)\,x\}$$
$$+4a^2\,(l^2+p^2)\,x^2-8arl\,(pq-l^2)\,y-4a^2\,(pq-l^2)^2-4r^2l^2\,(x^2+y^2)=0\ \ldots(\mathrm{F});$$

or putting ρ^2 for x^2+y^2,

$$r^2\rho^4+\{(a^2-r^2)(l^2+q^2)-2arly-2ar(r+p)x\}\rho^2$$
$$+4a^2rx(px+ly)+2arq\{l^2+q^2+2(pq-l^2)\}x$$
$$+2arl(l^2+q^2-2rq)y-a^2\{l^2+q^2+2(pq-l^2)\}^2=0.$$

The curve represented is evidently bicircular. If $l=0$, the equation becomes $\{r^2y^2+[rx-a(r+p)]^2\}(x^2+y^2-q^2)-4a^2pry^2=0$ ..... (G); and the curve has a double point at $y=0$, $x=a\frac{r+p}{r}$. From this it may at once be inferred that the curve represented by the more general equation has also a finite double point. For the description by the extremity of l is continuous or discontinuous in the same measure as the description by a point taken generally on the moving line. The curve is therefore the "pedal curve" of a conic section.

13. The equation (F) is also general. For we have at our disposal the three constants of the arbitrary circle, the two constants of the circle passing through the centre of the first circle together with p and l, making 7 constants in all.

The simpler equation (G) contains a large class of curves. We may write it in the form,

$$(\rho^2-\mu^2)\{y^2+(x-\nu)\}^2-\pi^2y^2=0.$$

If $\nu=0$, the origin is a double point, and we have

$$\rho^4=\mu^2x^2+(\pi^2+\mu^2)y^2,$$

representing the well known locus of the feet of the perpendiculars on the tangents of a conic section from its centre.

In order to obtain the Lemniscate of Bernouilli, we must make

$$p=-r,\quad a=r\sqrt{2}.$$

We are thus enabled to describe the curve continuously by very simple means.

In order that the equation should represent a Limaçon, we must have $r=p$ or $a=0$. Consequently we can only approximate to the curve. Figs. (9), (10) represent curves described in the above manner.

Here I leave the subject. Although the methods noticed appear at first sight very different in mechanical relation and result, they are nevertheless theoretically connected, as we have seen. Nor is the practical diversity so great as to prevent the combination of the processes in one apparatus. It is better not to interfere with this principle of unity, by discussing other graphical contrivances by means of extended strings, the intersections of conditioned lines or trochoidal motion; nor

could justice be done to the subject without unduly extending my limits.

The author exhibited some simple contrivances by which his figures had been drawn. The President made some remarks on the subject, more particularly bearing upon the *point*-description of such curves, whereas Mr. Roberts' constructions were confined to their *continuous* description.

Prof. Hirst next read a paper "On degenerate forms of Conics," in the course of which he referred to the Chairman's paper, "On the curves which satisfy given conditions."

The following presents were received:—

"Crelle's Journal," 69 Band, iii^{e} Heft.

"Annali di Matematica," Serie ii^{a}., tomo ii., fasc. i. and ii., Agosto 1868, Ottobre 1868.

"Memorie del Reale Istituto Lombardo," vol xi. della Serie iii., fasc. iv., v. ed ultimo, 1867.

"Mémoire de Géométrie pure sur les Surfaces du troisième ordre."

"Rendiconti adunanza," del 7 May, 1868, Serie ii., vol. iv. 1867, vol. i. 1868.

"Solenni adunanze," vol. i., fasc. iv.

"Royal Society's Catalogue of Scientific Papers," vol. ii., COA—GRA.

In the Proceedings of the Society, under date of 26th November, 1868,* it will be seen that Professor Sylvester gave an account of "Spirals whose arcs and the arcs of whose pedals are algebraically related"; the subject having grown under his hands, he made a further communication to the Society (March 11th, 1869), on a "Theorem in Logarithmic Waves," and went into details connected with the subject, and its bearing upon Cyclodes and the properties of Numbers. The Theory still continued to occupy the Author's mind (and from the importance of its results he has been led to style it his *capo d'opera*), so that he subsequently communicated to the Society (June 10th, 1869) his matured results in the following paper.†

* No. xvi., p. 118.

† The interesting character and importance of this paper, in addition to some delay having occurred with reference to other communications, and Professor Sylvester's desire to have it put into circulation, will perhaps sufficiently account for its thus appearing out of its proper position.

*Outline trace of the theory of Reducible Cyclodes**, i. e. *a particular Family of successive Involutes to a Circle whose determination depends on the solution of an Algebraico-Diophantine Equation, and of the Number and Classification of the forms of such Family for any given order of succession*†. By J. J. Sylvester, LL.D., F.R.S., of the Institute of France, the Royal Academies of Science of Berlin, Naples, Göttingen, &c. &c., Professor of Mathematics at the Royal Military Academy, Woolwich.

La conquête de la vérité exige le concours harmonique de toutes les facultés humaines.
Albert Réville.

(1) A cyclode is the continued [nth] involute of a circle. The centre of the circle is the pole of the cyclode.

(2) If x, y are the *inclination* and *polar distance* of the tangent to a cyclode, $y=(x, 1)^n$ is its equation, where n is the number of unwindings, or, so to say, the order of the involution. When the radius of the circle is unity and

* See Phil. Mag. (October and December 1868) for preliminary notions on involutes in general and on the successive involutes to a circle. Attached to the second of these notes is a valuable plate containing beautifully drawn specimens of cyclodes of the first three orders, executed by Mr. Spottiswoode.

† It may be right to mention that the results arrived at in the denumerational portion of this paper have been obtained almost exclusively by processes of observation, comparison, experiment, and induction, satisfying the sense of the perfect and beautiful; and consequently the theory must be regarded as essentially non-mathematical, it having been authoritatively laid down by a great naturalist and eminent public instructor in the 'Fortnightly Review' for June 1869, that Mathematics is "that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing" [does any science know much?] "of causation." Lagrange has expressed himself pointedly (in a passage which I cannot at the moment recall) in the very opposite sense to Professor Huxley; so certainly would Fermat, Euler, Gauss, Jacobi, Abel, Cauchy, Eisenstein, Kronecker, Plücker, Riemann, and the other great masters of our art, had their opinion on the subject been appealed to. In a subjective sense this paper owes its origin to my attention having been drawn by the Rev. James White, of Trinity Church, Woolwich, to Capt. Moncrieff's selfreversing-gun carriage, the rack in which for steadying and regulating the motion is the curve which would be traced on the plane of a wheel rolling on a rail by a point fixed on above or below the rail. When the point is level with the centre of the wheel, the curve traced is Archimedes's Spiral, when level with the rail the first Involute of the circle, and when midway between the rail and the centre a *Square root* (in the quaternion sense) of that Second involute to the circle whose apse is midway between the centre of the circle and the cusp of the first involute, to which second involute or cyclode I have given the name of the Norwich Spiral, to mark the fact of its birth at the Meeting of the British Association held last year in that city. The theory has since grown upwards and outwards faster than I have been able to climb after it, like the *bean-stalk* of profound significance in the well-known child's story. A very elegant and simple (because philosophically conceived) solution of the problem of finding the form of the rack (which I call a *Convolute* to the circle, and which the Arsenal people mistakenly but naturally enough supposed to be a Cycloidal curve) was given by Mr. White in the 'Educational Times' (*vide* Reprint for latter half of 1868). The problem was originally proposed many years ago by Mr. Earnshaw, as a Senate-House problem, suggested to him by seeing a boy trundle a wheel against a brick wall in the streets of Cambridge.

the axis of reference is the line joining the pole to the cusp of the $(n-1)$th evolute of the cyclode (*i. e.* the first involute of the generating circle), the coefficients of x^n, x^{n-1} become 1 and 0 respectively. In what follows, it is to be understood that in general this simplified form is the one referred to.

(3) If r is radius, ρ radius of curvature, s length of arc reckoned from any origin on the curve,

$$r^2=y^2+\left(\frac{dy}{dx}\right)^2,$$

$$s=\int dx\rho=\int dxy+\frac{dy}{dx}.$$

Hence every cyclode is rectifiable, s and r^2 being unicursally related.

(4) The arco-radial equation is of the form $F(s,r^2)=0$. When $y^2+y'^2=\square$, s and r become unicursally related, so that $F(s,r^2)=\Phi(s,r)\Phi(s,-r)$, and the cyclode is said to be *reducible**.

(5) Order is the degree of the parabolic function $(x,1)^n$. Class is the number of distinct factors in $(x,1)^n$, *i. e.* the number of *distinct* tangents that can be drawn to a cyclode from its pole; for in the case of such tangents $y=0$.

(6) Reducible cyclodes form a *family*: the *class* and the *order* of every reducible cyclode must both be *even*. They are of two *kinds*, symmetrical and unsymmetrical. The simplified form for the symmetrical kind is $(x^2,1)^\nu$. A symmetrical cyclode is divided into two equal and similar parts by the line joining the pole to the cusp of its $(n-1)$th evolute. Non-symmetrical reducible cyclodes are left- and right-handed; they appear in pairs, the equations to the two of a pair being $y=Fx$, $y=F(-x)$ respectively.

(7) Let ν be divided in all possible ways (say ω ways) into a given number μ of parts; let the *partitionments* be $P_1, P_2, \ldots P_\omega$. Then $(P_1, P_2, \ldots P_\omega)^2$ developed contains terms of the form $P_j . P_j$, and of the form $P_j P_k$, $P_k P_j$. Every such binary combination may be called a *Diptych* of a group of reducible cyclodes of order 2ν and class 2μ; and the total of such groups will be the total of reducible cyclodes of order 2ν and class 2μ. The sum of these totals for all classes from 2 to 2ν will be the *ensemble* of reducible cyclodes of order 2ν.

(8) The *alæ* of a diptych are the two sets of terms which form its two sides. These *alæ* will sometimes be represented as folded together, and sometimes expanded, as convenience may dictate. Under these *alæ* each diptych, as will be seen, shelters a group or brood of cyclodes. The relation between the *alæ* and the parabolic functions $(x,1)^n$ of the reducible cyclodes proper to the *diptych* is as follows. Suppose that

$$i, j, k, \ldots \quad | \quad i_1, j_1, k_1, \ldots$$

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* σ (the length of the arc of the pedal in respect of the pole) is $\int dx . r$: so that s, σ, x, y, r are all unicursally related in the case of *reducible* cyclodes. The equation in the text, $y^2+\left(\frac{dy}{dx}\right)^2$ = a perfect square, is the algebraico-diophantine equation referred to in the title. A more general equation, towards the solution of which I have made a few steps, is the following— $y^2+y'^2=\eta^2+\lambda\eta'^2$; but I do not at present perceive its geometrical bearing.

is the diptych. Then by hypothesis

$$i+j+k\ldots=i_1+j_1+k_1\ldots=\nu,$$

the semi-order.

$i, j, k\ldots$ are the same in number (that number being μ the semi-class) as $i_1, j_1, k_1, \ldots$

$y=UU_1$* is the equation to every cyclode proper to the given diptych,

where $U=(x+a)^i\,(x+b)^j\,(x+c)^k\ldots$

$$U_1=(x+a_1)^{i_1}(x+b_1)^{j_1}(x+c_1)^{k_1}\ldots$$

To find $a, b, c, \ldots; a_1, b_1, c_1, \ldots$ write

$$V=(\lambda+a)(\lambda+b)(\lambda+c)\ldots, \qquad V_1=(\lambda+a_1)(\lambda+b_1)(\lambda+c_1)\ldots,$$

$$E=\left(2i\frac{d}{da}+2j\frac{d}{db}+2k\frac{d}{dc}\ldots\right), \quad E_1=\left(2i_1\frac{d}{da_1}+2j_1\frac{d}{db_1}+2k_1\frac{d}{dc_1}\ldots\right),$$

and let $EV=E_1V_1=V-V_1$ for all values of λ. This implies

$$2\Sigma i=2\Sigma i_1=\Sigma a-\Sigma a_1,$$

$$\Sigma 2i(b+c\ldots)=\Sigma 2i_1(b_1+c_1+\ldots)=\Sigma ab-\Sigma a_1b_1,$$

$$\Sigma 2i(bc\ \ldots)=\Sigma 2i_1(b_1c_1\ \ \ldots)=\Sigma abc-\Sigma a_1b_1c_1,$$

. . . . . .

. . . . . .

The equality $\Sigma i=\Sigma i_1$ has been presupposed between the elements of the diptych. Remain $2\mu-1$ equations between the 2μ quantities $a, b, c, \ldots$ $a_1, b_1, c_1, \ldots$

Hence the quantities themselves are, as they should be, indeterminate; but their differences will be determinate (as is easily proved)†. If we employ the simplified form, then in addition to the $2\mu-1$ equations above given, there will be the equation

$$\Sigma a+\Sigma a_1=0,$$

which will make the system perfectly determinate.

(9) If $i, j, k, \ldots i_1, j_1, k_1$ are all unequal, the number of solutions will be

$$1\times1\,.\,2\times2\,.\,3\ldots\times(\mu-1)\,.\,\mu,$$

i. e. $\pi(\mu-1)\pi\mu$.

(10) The effect of interchanging the *alæ* of the diptych (when unsymmetrical) is to convert every a into $-a$, and every a_1 into $-a_1$; but when the *alæ are alike* this change leaves the solutions unaltered. The solutions proper to a conformable diptych (*i. e.* one in which the alæ agree term for term) resolve themselves into two groups:—one group in which

$$a=-a_1, \quad b=-b_1, \quad c=-c_1, \ldots,$$

which is the group of symmetrical reducible cyclodes; and a second

* The U and U_1 may be called the two alar constituents, or more simply the segments of the parabolic function appertaining to any reducible cyclode; and we see that the radial tangents (or tangent radii from the centre) of every such cyclode constitute two distinct groups corresponding to these segments.

† The property of *reducibility* is of course unaffected by the rotation of a cyclode round the pole: hence, not the roots of the parabolic function, but their differences ought to determine the existence of this property.

group, in which the solutions appear in *pairs* of the form

$$\begin{array}{cccccccc} a, & b, & c, \ldots & a_1, & b_1, & c_1, \ldots \\ -a_1, & -b_1, & -c_1, \ldots & -a, & -b, & -c, \ldots \end{array}$$

Thus non-symmetrical reducible cyclodes are of two sorts:—congeminate, where the left- and right-handed curves belong to the same system of equations; and contra-geminate, where the left- and right-handed curves belong to opposite systems. The former are associated with the symmetrical species, inasmuch as they belong to diptychs which engender along with them the symmetrical species; the latter belong to unconformable diptychs, *i. e.* with dissimilar *alæ**. We now see more clearly the exactitude of the representation before stated of the diptychs of a given class by $(P_1, P_2, \ldots P_\omega)^2$, and the meaning of the coefficient 2 which affects the product $P_j . P_k$.

(11) The equation-system for unsymmetrical reducible cyclodes may be said to be *dualistic*. For symmetrical cyclodes it is *monistic*†, and takes the form

$$\begin{aligned} \nu \quad &= \Sigma a, \\ \Sigma i(b+c+d\ldots) \quad &= 0, \\ \Sigma i(bc+bd+cd\ldots) &= \Sigma abc, \\ \Sigma i(bcd\ldots) \quad &= 0, \\ \Sigma i(bcde\ldots) \quad &= \Sigma abcde, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned}$$

Hence if $i, j, k, \ldots$ are all unlike, the number of solutions for a monistic system belonging to a diptych of the μth class‡ is the product of μ terms of the fluctuating progression

$$1 . 1 . 3 . 3 . 5 \ldots$$

(12) The number of solutions of a determinate algebraic system of equations may be called its denumerant. The denumerant may be algebraical or arithmetical.

In estimating the former, all solutions count, whether or not deducible from one another by interchange between the unknowns.

In estimating the latter, solutions which become identical by permuting

* The equation-system belonging to a conformable diptych may be resolved into two distinct systems containing respectively the symmetrical and the non-symmetrical solutions.

† The *dualistic* system for a conformable diptych resolves itself into the *monistic* system above given, which contains the symmetrical solutions, and another system (with which I do not think it desirable to encumber this sketch, but) which I have easily succeeded in isolating, and which contains the congeminate solutions. The congeminate denumerants do not, however, appear to lend themselves to a distinct law of summation apart from the general mass as the symmetrical ones do.

‡ It will be convenient to refer to the diptych of a cyclode of the order 2ν and class 2μ as being itself of the order ν and class μ.

the unknowns are regarded as one and the same solution. Thus, *ex. gr.*, the algebraical denumerant of $|x^2+y^2=c, x^4+y^4=e|$ is 8, its arithmetical denumerant is 4.

In the theory under consideration, it is always the arithmetical* denumerant which is intended, unless the contrary is expressed. Thus we see that one cause of the *Reduction* of the denumerant of a diptychical system of equations is the *consolidation* of several solutions into a single one. A second cause of reduction, arising from roots passing off to infinity, may be termed *evaporation*. *Ex. gr.*, in the equation-system $|x-ky=c, x^3-y^3=e|$ the denumerant is 3; but when $k=1$, one root passes off to infinity and the denumerant becomes 2.

Consolidation and evaporation may go on simultaneously: thus, *ex. gr.*, in the system $|x+ky=c, x^3+y^3=e|$, when $k=1$, the denumerant is reduced first by evaporation to 2, and then by consolidation to 1.

(13) The problem of which I shall indicate a complete method of solution is that of obtaining the denumerant to any given diptych whatever.

The original problem proposed for solution was to find the *total* number of reducible cyclodes of any given order; not only this, but the number of the same of any given *class*, as well as *order*, admits of easy *statement*.

If $D(n, m)$ be used to denote the total number of reducible cyclodes of the order $2n$ and class $2m$ (obtainable from a given circle), then

$$D(r+s, r)=D(r+s, s)=\frac{\Pi(r+s-1)\Pi(r+s-2)}{\Pi r\Pi(r-1)\Pi s\Pi(s-1)}.$$

Calling $D(r+s, r)=Q_{r,s}$, the annexed Table exhibits the value of $Q_{r,s}=Q_{s,r}$† for different values of r and s.

* It is a singular and advantageous fact, quite contrary to what might have been expected from analogy, that it is these denumerants (corresponding to the actual number of distinct geometrical forms) and not the algebraical denumerants (the ordinary measures of the so-called orders of the various systems of equations) which lend themselves to summation. In the common theory of partitions *per contra*, if we wish to express their number algebraically, we must reckon permuted as distinct arrangements; this is as if we should substitute algebraical for arithmetical denumeration in the theory under consideration. This fact is partly to be accounted for by considering that the diophantine equation to be solved, viz. $f^2+f'^2=\phi^2$, may be attacked *exoscopically*, the coefficients in f and ϕ being posited as the unknowns, in which scheme of procedure permutations of equal factors in f would not present themselves at all: my original tentative efforts were made in this the seemingly most natural direction to follow; but how futile it would have been to have persevered in this course, and how all but impossible to disentangle, by aid of it, the classes and genera and diptychical groups, is apparent from the very nature of the results here brought to light. Of all the analytical questions left unsolved by my predecessors or contemporaries which I have yet grappled with (and they are neither few nor facile), this presents beyond all comparison the hardest knot I have ever succeeded in untying. But what will not yield to patient contemplation! Nature, though coy, is kind, and does not predetermine to mock the pursuit of her worshippers.

† That $Q_{r,s}=Q_{s,r}$ implies $D(n, r)=D(n, n-r)$,—a truly remarkable theorem, which exists also when Δ replaces D.

r \ s	Q	1	2	3	4	5	6
	1	1	1	1	1	1	1
	2	1	3	6	10	15	21
	3	1	6	20	50	105	196
	4	1	10	50	175	498	1176
	5	1	15	105	496	1764	5292
	6	1	21	196	1176	5292	19404

$\mathrm{D}(m, n)$ will consist of a certain number of symmetrical cyclodes, say $\Delta_{m,n}$, and $\frac{\mathrm{D}(m,n)-\Delta(m,n)}{2}$ pairs of unsymmetrical cyclodes, giving a total of $\frac{\mathrm{D}(m,n)+\Delta(m,n)}{2}$ absolutely distinct forms when left- and right-handed similar curves are treated as identical.

(14) Let

$$\Delta(r+s, r)=q_{r,s},$$

then $q_{r,s}(=q_{s,r})$ is the quantity in the rth line and sth column of the Table B subjoined, deduced by zigzag multiplication from the adjacent Table A*.

TABLE A.

1	1	1	1	1	1
1	2	3	4	5	6
1	2	3	4	5	6
1	3	6	10	15	21
1	3	6	10	15	21
1	4	10	20	35	56

TABLE B.

r \ s	q†	1	2	3	4	5	6
	1	1	1	1	1	1	1
	2	1	1	2	2	3	3
	3	1	2	4	6	9	12
	4	1	2	6	9	18	24
	5	1	3	9	18	36	60
	6	1	3	12	24	60	100

* Calling

$$\left.\begin{array}{l} 1,\ 1,\ 1,\ 1,\ 1,\ \ldots = \mathrm{L} \\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots = \mathrm{L}_1 \\ 1,\ 3,\ 6,\ 10,\ 15,\ \ldots = \mathrm{L}_2 \\ 1,\ 4,\ 10,\ 20,\ 35,\ \ldots = \mathrm{L}_3 \\ \ldots\ldots\ldots \end{array}\right\},$$

the Table A itself is the product of the zigzag multiplication of

$$\left\{\begin{array}{cccccc} \mathrm{L} & \mathrm{L}_1 & \mathrm{L}_2 & \mathrm{L}_3 & \mathrm{L}_4 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \end{array}\right\}.$$

† It is easily seen that when

$$r=2h+1,\ s=2k+1;\ q_{r,s}=\left(\frac{\Pi(h+k)}{\Pi h\Pi k}\right)^2,$$

$$r=2h,\quad s=2k+1;\ q_{r,s}=\frac{\Pi(h+k)\Pi(h+k-1)}{\Pi h\Pi(h-1)\cdot(\Pi k)^2},$$

$$r=2h,\quad s=2k;\ q_{r,s}=\frac{(\Pi(h+k-1))^2}{\Pi(h-1)\Pi h\Pi(k-1)\Pi(k)}.$$

(15) $D(n)$, $\Delta(n)$ being used to denote the absolute number of reducible cyclodes and the number of symmetrical reducible cyclodes, respectively, of the order $2n$ of all classes, we have

$$D(n) = \frac{\Pi(2n-2)}{\Pi n \,.\, \Pi(n-1)},$$

$$\Delta(2k+1) = \frac{\Pi(2k)}{(\Pi k)^2},\ \Delta(2k) = \tfrac{1}{2}\Delta(2k+1).$$

The total number of reducible cyclodes and of reducible symmetrical cyclodes for all orders from 2 to 20 inclusive is exhibited in the Table below, where n is the semi-order.

n	Δ	D
1	1	1
2	1	1
3	2	2
4	3	5
5	6	14
6	10	42
7	20	132
8	35	429
9	70	1430
10	125	5236

The ratio of the successive values of Δ alternates between 2 and a variable quantity which converges ascendingly towards 2; that of the successive values of D converges (also ascendingly) towards 4. It will be observed that D and Δ agree, until $2n=8$, showing that the non-symmetrical reducible cyclodes only begin to make their appearance with the eighth order of involution, *i. e.* after the generating tight string has been unwound 8 times in succession.

(16) The *number of distinct species of diptychs of any given* class *is limited,* being independent of the order; *and every diptych of such species will have the same denumerant.* The doctrine of *genus and species* is founded on the notion of *parallel equalities**.

* The foundation of the *conception* of the doctrine of parallel equalities lies in the easily demonstrable fact that the leading coefficient in the final resolving equation of the *monistic* system of equations proper to any conformable diptych $(a\ b\ c\ldots l)^2$ is made up exclusively of powers of linear factors of the form $a-b$, $a+b-c-d$, $a+b+c-d-e-f$, &c., and that the same for the *dualistic* system proper to a general diptych $[a\ b\ c\ldots l \mid \alpha\beta\ \gamma\ldots\lambda]$ is made up exclusively of powers of linear factors of the form $a-\alpha$, $a+b-\alpha-\beta$, $a+b+c-\alpha-\beta-\gamma$, &c.: this fact is sufficient for proving that evaporation can only commence to take place by virtue of the existence of one or more parallel equalities, but is not sufficient to demonstrate the truth of the grand law that not only the *existence,* but the *amount* of evaporation is governed solely and exclusively by the nature of the parallel equality or system of parallel equalities (when several coexist either separately or

A parallel equality is one existing between the sum of any number of elements in one ala of a diptych, and the sum of a like number of elements in the opposite ala; *i. e.*, it is an inter-alar equinomial relation of equality to which it will sometimes be convenient to refer to as a *parallelism*. Two antagonistic parties of creditors to a bankrupt's estate, equal in number and claiming for equal amounts, gives an image of the parallelism in question. By the definition of a diptych there is necessarily one such parallel equality, viz. between all the elements in one *ala* and all in the other *.

Any two diptychs in which the entire system of parallel equations is the same for each are said to belong to the same *genus*, and their *algebraical* denumerants will be the same. When in two diptychs belonging to the same genus all the equalities between the elements of each *ala* taken alone in the one are matched by like equalities between the analogous elements of the corresponding *ala* in the other, they are said to belong to the same *species*, and their *arithmetical* denumerants will be identical†.

(17) Now let it be proposed to determine the denumerants, for greater simplicity say the total denumerants (including non-symmetrical as well as symmetrical cyclodes), of the several species belonging to any given class. There are as many unknowns, say, $d_1, d_2, \ldots d_s$, as there are species (s) included in such class: the total number of diptychs of that class belonging to any given order increases with the order, being represented by $(P_1, P_2, \ldots P_\omega)^z$, where $P_1, P_2, \ldots P_\omega$ denote the partitionments of the number n into m parts. For every assumed value of n we shall be able to obtain the value of of $D(P_1, P_2, \ldots P_\omega)^z$ as a known linear function of $d_1, d_2, \ldots d_s$; but $D(P_1, P_2, \ldots P_\omega)^z$ is a known function of n and m‡; hence we may obtain the values of asufficient number of linear functions of $d_1, d_2, \ldots d_s$ to obtain each of them separately; and we know *à priori* that each is integer and positive.

Thus, regarding the denumerants as gas in relation to the equation-systems viewed as the gross matter from which this gas is extracted (a simile suggested by Mr. Clifford), we witness in this theory the interesting phenomenon of this gas becoming again condensed, as it were (like solidified carbonic acid gas), into a sort of crystallized matter finer than that from which it was originally generated, the denumerants of the original equation-system becoming the unknowns of a new (linear) equation-system.

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mutually superimposed) in any given diptych. For a proof of what is stated in this note in respect of the leading coefficient of the monistic system, see Phil. Mag. for May 1869; I am also in possession of the corresponding proof and theory for the dualistic system.

* Consequently any partial inter-alar parallel equality implies a second complementary one; in other words, all the parallel equalities, excepting the implied total one, are conjugate and make their appearance in pairs.

† Thus we see that the elements of the diptych being all increased or diminished by the same quantity, or, more generally, being replaced respectively by the same linear function of each of them, will not affect the genus or species, as such substitution will leave the identities and parallel equalities unaltered; so that we may say that the species of a Reducible Cyclode is, as it were, a semiinvariant of the indices of the factors of its parabolic function.

‡ See art. 13, where $r = m$, $s = n - m$, and $D(r+s, r)$ means the same as $D(P_1, P_2 \ldots P_\omega)^z$ in the text above.

[Chasles's theory (with Zeuthen's, Cayley's, and Salmon's extensions) of conics which satisfy given conditions, the new theory of geometric Unicursal or successive Quadric Transformation, and this theory of Cyclodes, all belong to a peculiar branch of algebra to which might be given the name of pneumatic analysis; and they have many features in common.]

In the same way it may obviously be shown that the denumerants of the *symmetrical* cyclodes proper to the distinct species of conformable diptychs of any given class may be deduced from the solution of a system of linear equations as many in number as such species. In fact, however, only as many linear functions (in either case) need to be formed as there are distinct *genera*; for the denumerants (arithmetical) of the several *species* of the same genus are known, *à priori*, in terms of each other, since their algebraical denumerants are identical.

(18) A most important and wonderful relation exists between $D(r+s, r)$ or $\Delta(r+s, r)$ and the corresponding denumerants of the conformable diptych $[a^r, b^s]^2$, where a^r means a, r times repeated, and b^s means b, s times repeated. [Observe that any such diptych whose class $r+s$ is given belongs to a single genus; for we cannot have

$$\rho a+\sigma b=\rho' a+\sigma' b \text{ and } \rho+\sigma=\rho'+\sigma',$$

unless $a=b$, which is of course excluded.] Viz. *the total denumerant of this diptych is equal to* $D(r+s, r)$, *and its special denumerant is equal to* $\Delta(r+s, r)$.

(19) A very good example of *evaporation* is afforded by the diptych $[a^r]^2$; for the algebraical denumerant without evaporation would be $\Pi r \,.\, \Pi(r-1)$; but the actual algebraical denumerant must contain $(\Pi r)^2$. Hence the evaporation is *total*, *i. e.* there is no reducible cyclode of any order or class whose parabolic function is made up of distinct factors raised all to the same power (and, as a particular case, made up exclusively of distinct simple factors), except when the cyclode is of the second degree, for $\Pi r \,.\, \Pi(r-1)$ contains $(\Pi r)^2$ when $r=1$; and in fact the equation of the Norwich spiral, which gave birth to this theory, is in its simplified form $y=(x+1)(x-1)$. This is analogous to the multiplicity (in Hirst's and Cremona's theory) of the *principal points* of any nucleus (as such system is termed by Professor Cayley) when, and not before, the degree of the transformation transcends the second. In the present theory we see that, except for the Norwich spiral, in all reducible cyclodes a certain number of the points, the tangents to which pass through the pole, must be cusps of a more or less elevated order.

(20) As an example of the method of classification, grounded on parallel equalities, I subjoin the complete system of genera and species of diptychs of the third (corresponding to cyclodes of the 6th) class, with their corresponding denumerants*.

* The *algebraical* denumerant, it will be noticed, is the same for all the species of the same genus. Thus, *ex. gr.*, in genus (1) overleaf for the first species this is 12, for the second $\Pi 2 \,.\, 6$, for the third $\Pi 3 \,.\, 2$, for the fourth $\Pi 2 \,.\, \Pi 2 \,.\, 3$, and for the last $\Pi 2 \,.\, \Pi 3 \,.\, 1$. The first genus, it will be seen, contains four, the second and fifth two, the third, fourth, sixth, and seventh one *species* respectively.

Genus (1)	Denumerant.	Genus (2).	Denumerant.	Genus (5).	Denumerant.
$a\ b\ c$ $\alpha\ \beta\ \gamma$	12	$a\ b\ c$ $a\ \beta\ \gamma$	10	$a\ b\ c$ $a\ a\ \gamma$	4
$a\ b\ c$ $\alpha\ \beta\ \beta$	6	$a\ b\ c$ $a\ \beta\ \beta$	5	$a\ b\ b$ $a\ a\ \gamma$	2
$a\ b\ c$ $\alpha\ \alpha\ \alpha$	2	Genus (3)*.		Genus (6).	
$a\ b\ b$ $\alpha\ \alpha\ \beta$	3	$a\ b\ c$ $a\ b\ c$	7	$a\ b\ c$ $a\ a\ a$	-
$a\ b\ b$ $\alpha\ \alpha\ \alpha$	1	Genus (4).		Genus (7).	
		$a\ b\ b$ $a\ b\ b$	1	$a\ a\ a$ $a\ a\ a$	0

Of the above 7 genera, the 3rd and 4th alone give rise to symmetrical cyclodes, the Δ (*i. e.* the symmetrical) denumerant for the 3rd being 3 and for the 4th 1.

(21) All species of conformable diptychs necessarily belong to distinct genera; *i. e.* the genera of conformable diptychs each contain only a single species. The conformable species of the fourth† class are as follows: [the numbers under Δ signify the number of *symmetrical* cyclodes proper to each genus, and under E the amount of evaporation.]

Species.	Δ.	E‡.
$(a\ b\ c\ d)^2$	9	0
$(a\ b\ b\ d)^2$	4	1
$(a\ a\ c\ c)^2$	1	5
$(a\ b\ b\ b)^2$	1	3
$(a\ a\ a\ a)^2$	0	9
$(a\ b\ c\ b+c-a)^2$	7	2
$(a\ b\ b\ 2b-a)^2$	3	3

* The *special* denumerant means the number of *symmetrical* cyclodes contained by a diptych. The diptych $(a\ b\ c)^2$ having the special denumerant 3 and the total denumerant 7, we see that the number of congeminate pairs is $\frac{7-3}{2}$, *i. e.* 2. So the number of congeminate *pairs* to $(a\ b\ c\ d)^2$ is 36; and I have almost conclusive grounds for believing that in general the value of $D-\Delta$ for $(a_1, a_2, \ldots a_i)^2$ is $(i-2)(\Pi(i-1))$; so that the number of right- and left-handed (congeminate) pairs contained in such diptych is

$$\frac{(i-2)(\Pi(i-1))^2}{2}.$$

† There are upwards of 40 distinct *genera* of the 4th class when unconformable diptychs are taken into account. It is a problem of great interest to determine the number of genera and species contained in any given class.

‡ It should be stated that the E column is deduced from the Δ column, and not *vice*

(22) The conformable species of the fifth class will be most easily represented concretely by means of actual numbers, so as to avoid the necessity of employing equations to denote the parallel equalities *.

Types of species.	Δ.	E.
$(1\ 2\ 3\ 5\ 8)^2$	45	0
$(1\ 2\ 3\ 5\ 5)^2$	21	3
$(1\ 2\ 3\ 3\ 3)^2$	6	9
$(\dot{1}\ \dot{2}\ \dot{2}\ 2\ \dot{3})^2$	5	15
$(1\ 2\ 2\ 5\ 5)^2$	9	9
$(1\ 1\ 2\ 2\ 2)^2$	2	21
$(1\ 1\ 1\ 1\ 2)^2$	1	21
$(1\ 1\ 1\ 1\ 1)^2$	0	45
$(\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ 8)^2$	43	2
$(\dot{1}\ \ddot{2}\ \ddot{4}\ \ddot{5}\ \acute{7})^2$	41	4
$(\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ \dot{5})^2$	39	6
$(1\ \dot{3}\ \dot{4}\ \dot{4}\ \dot{5})^2$	20	5
$(\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ 4)^2$	19	7
$(\ddot{1}\ \dot{2}\ \ddot{2}\ \ddot{3}\ \acute{4})^2$	18	9
$(\dot{1}\ \dot{2}\ \dot{2}\ \dot{3}\ 3)^2$	8	13

versâ. It might have been imagined, *à priori*, that the amount of evaporation could be determined independently, in the first instance, and the reduction of the denumerants inferred therefrom; but it appears as a datum of experience that the formulæ of denumeration are always simpler than those of evaporation, although, of course, theoretically speaking, the evaporating solutions ought to be obtainable by calculating the solutions for which the new homogenizing variable takes the value zero, or, which means the same thing, for which $1=0$. In a word, although in the order of nature the evaporation is the cause of the reduction, in the order of our knowledge the reduced denumerant, coming first, plays the part of a subtractor, and the evaporation that of a remainder.

* Since the above was sent to press, I have hit upon a very simple expedient for representing the parallels of any form of diptych by means of marks of connotation. This method is used in the sequel; but I have not thought it worth while to disturb the text in this place otherwise than by introducing marks of connotation to distinguish the terms of the parallel equalities. It ought to be noticed that when any element of a parallel equality occurs more than once, the mark of connotation is to be placed indifferently over any (but only one) of the group of quantities thus repeated.

I may take advantage of the blank space left at the foot of this page to refer, although a little out of order, to Mr. Clifford's extremely ingenious idea of making the proof of the existence of a root to an algebraical function (fx) of degree n depend on the same for the degree $\frac{n(n-1)}{2}$, and consequently in the last resort on the same for a function of an odd degree, for which case it flows direct from the principle of continuity. Equating

(23) Call the cyclodic order 2Ω, and the class 2χ.

Then for $\Omega=1, 2, 3, 4, 5, 6, 7$ the diptychs grouped in classes, omitting the *vacuous* ones [*i. e.* those for which the evaporation is *total*], will be as follows:—

$\Omega=1$ $\chi=1$ $(1)^2$

$\Omega=2$ $\chi=1$ $(2)^2$

$\Omega=3$ $\chi=1$ $(3)^2$
$\chi=2$ $(1\ 2)^2$

$\Omega=4$ $\chi=1$ $(4)^2$
$\chi=2$ $(1\ 3,\ 2\ 2)^2$
$\chi=3$ $(1\ 1\ 2)^2$

$\Omega=5$ $\chi=1$ $(5)^2$
$\chi=2$ $(1\ 4,\ 2\ 3)^2$
$\chi=3$ $(1\ 1\ 3,\ 1\ 2\ 2)^2$
$\chi=4$ $(1\ 1\ 1\ 2)^2$

$\Omega=6$ $\chi=1$ $(6)^2$
$\chi=2$ $(15, 24, 23)^2$
$\chi=3$ $(1\ 1\ 4,\ 1\ 2\ 3,\ 2\ 2\ 2)^2$
$\chi=4$ $(1\ 1\ 1\ 3,\ 1\ 1\ 2\ 2)^2$
$\chi=5$ $(1\ 1\ 1\ 1\ 2)^2$

$\Omega=7$ $\chi=1$ $(7)^2$
$\chi=2$ $(1\ 6,\ 2\ 5,\ 3\ 4)^2$
$\chi=3$ $(1\ 1\ 5,\ 1\ 2\ 4,\ 1\ 3\ 3,\ 2\ 2\ 3)^2$
$\chi=4$ $(1\ 1\ 1\ 4,\ 1\ 1\ 2\ 3,\ 1\ 2\ 2\ 2)^2$
$\chi=5$ $(1\ 1\ 1\ 1\ 3,\ 1\ 1\ 1\ 2\ 2)^2$
$\chi=6$ $(1\ 1\ 1\ 1\ 1\ 2)^2$.

Expanding we obtain the following set of diptychs, the numbers opposite

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to zero each coefficient in the linear remainder of fx divided by x^2+px+q, we obtain two simultaneous equations of known degrees (dependent on the value of n) in p and q. Now the pinch of the required proof consists in our being able to show that the p, q system satisfying these equations has exactly $\frac{n(n-1)}{2}$ values, neither more nor less; so that it is requisite to establish, *à priori*, a certain definite amount of evaporation, which I believe no one has yet quite succeeded in doing. This example affords an instructive corroboration of the remark in the last preceding foot-note, concerning evaporation coming second in the order of our knowledge, although first in the order of nature. The arrangement of the diptychical denumerants for a given order and class in the text which follows is columnar. The more natural arrangement (and one so likely to be fruitful in consequences, that I much regret not thinking of it in time) is obviously that of a square, of which each side will contain as many places as there are partitionments corresponding to the order and class. In that diagonal which separates the square into two symmetrical sets of figures (and nowhere else) there will be double entries, corresponding to the D and Δ denumerants respectively of the conformable diptychs.

to which are their symmetrical and total denumerants respectively. The Table extends from $\Omega=1$ to $\Omega=7$ inclusive.

Diptych.	Δ.	D.
$(1)^2$	1	1
$(2)^2$	1	1
$(3)^2$	1	1
$(1, 2)^2$	1	1
$(4)^2$	1	1
$(1, 3)^2$	1	1
1 3 . 2 2	0	1
$(2\,2)^2$	0	0
$(1\,1\,2)^2$	1	1
$(5)^2$	1	1
$(1\,4)^2$	1	1
(1 4 . 2 3)	0	2
$(2\,3)^2$	1	1
$(1\,1\,3)^2$	1	1
(1 1 3 . 1 2 2)	0	2
$(1\,2\,2)^2$	1	1
$(1\,1\,1\,2)^2$	1	1
$(6)^2$	1	1
$(1\,5)^2$	1	1
(1 5 . 2 4)	0	2
$(2\,4)^2$	1	1
(1 5 . 3 3)	0	1
(2 4 . 3 3)	0	1
$(3\,3)^2$	0	0

Diptych.	Δ.	D.
$(1\,1\,4)^2$	1	1
(1 1 4 . 1 2 3)	0	4
$(1\,2\,3)^2$	3	7
1 1 4 . 2 2 2	0	1
1 2 3 . 2 2 2	0	1
$(2\,2\,2)^2$	0	0
$(1\,1\,1\,3)^2$	1	1
(1 1 1 3 . 1 1 2 2)	0	4
$(1\,1\,2\,2)^2$	1	1
$(1\,1\,1\,1\,2)^2$	1	1*
$(7)^2$	1	1
$(1\,6)^2$	1	1
(1 6 . 2 5)	0	2
$(2\,5)^2$	1	1
(1 6) (3 4)	0	2
(2 5 . 3 4)	0	2
$(3\,4)^2$	1	1
$(1\,1\,5)^2$	1	1
(1 1 5 . 1 2 4)	0	4
$(1\,2\,4)^2$	3	7
(1 1 5 . 1 3 3)	0	2
(1 2 4 . 1 3 3)	0	5
$(1\,3\,3)^2$	1	1
(1 1 5 . 2 2 3)	0	3
(1 2 4 . 2 2 3)	0	4
(1 3 3 . 2 2 3)	0	2
$(2\,2\,3)^2$	1	1

* Thus for the 12th order and 4th class the number of symmetrical reducible cyclodes is $1+1$, *i. e.* 2, and the total number is $1+2\,.\,2+1+2\,.\,1+2\,.\,1+0=10$; so for the 12th order and 8th class these numbers are $1+1$ and $1+2\,.\,4+1$, *i. e.* 2 and 10 as before, as

Diptych.	Δ.	D.
$(1\,1\,1\,4)^2$	1	1
$(1\,1\,1\,4.1\,1\,2\,3)$	0	6
$(1\,1\,2\,3)^2$	4	16
$(1\,1\,1\,4.1\,2\,2\,2)$	0	3
$(1\,1\,2\,3.1\,2\,2\,2)$	0	7
$(1\,2\,2\,2)^2$	1	1
$(1\,1\,1\,1\,3)^2$	1	1
$(1\,1\,1\,1\,3.1\,1\,1\,2\,2)$	0	4
$(1\,1\,1\,2\,2)^2$	2	6
$(1\,1\,1\,1\,1\,2)^2$	1	1*

The above Table gives a complete conspectus of the groups of reducible cyclodes of all the even (the only admissible) orders up to the 14th inclusive.

(24) The diophantine equation $y^2+y'^2=\square$ may be transformed into $1+\left(\frac{y'}{y}\right)^2=\square$.

In this equation we may suppose

$$y=(x+a)^{\alpha}.(x+b)^{\beta}(x+c)\ \ldots\ldots,$$

$\alpha, \beta, \gamma \ldots$ being now *general* quantities and no longer necessarily integers.

If $\eta=\log y=\Sigma\alpha \log (x+a)$,

x, η are the coordinates of a compound logarithmic wave.

If the number of terms in Σ is even, and the quantities $a, b, c \ldots$ be divided arbitrarily into two sets containing a like number of terms, as, say, $a, b, c \ldots, a', b', c' \ldots, \alpha, \beta, \gamma \ldots, \alpha', \beta', \gamma'$ being corresponding values of the coefficients,

$$1+\left(\frac{d\eta}{dx}\right)^2=\square\text{, provided that}$$

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should be the case according to article (13). The equations to be resolved are of the degree D for unconformable, and of the degrees Δ and $\frac{D-\Delta}{2}$ for conformable diptychs; hence up to the 12th order of reducible cyclodes, inclusive, we have not to deal with equations of a higher degree than the 4th, and consequently all the reducible cyclodes of order inferior to 14 admit of explicit algebraical representation.

* Looking at the 3rd and 4th classes, the sums of the numbers in the D column will be found to be 30 and 34 respectively; but on doubling the numbers printed opposite to the unconformable diptychs (which represent not single but paired cyclodes), the number is the same (as it ought to be) for each class, viz. 50, that being the number in the 3rd column and 4th line (or *vice versâ*) of the Q Table art. 13. Similarly the corresponding number for the 2nd and 5th classes, doubling as before, becomes 15 instead of 9 for the one, and 11 for the other. It is a noticeable fact that the number of pairs is not the same in two conjugate classes.

$$2\alpha=\frac{(a-a')(a-b')(a-c')\ldots.}{(a-b)(a-c)}, \quad -2\alpha'=\frac{(a'-a)(a'-b)(a'-c)\ldots.}{(a'-b')(a'-c')\ldots.}\ *.$$

and similarly for $\beta, \gamma \ldots.; \beta', \gamma' \ldots.$

From these equations it follows that $\Sigma\alpha=\Sigma\alpha'$ as before.

(25) Conversely, provided

$$\alpha+\beta+\gamma\ldots=\alpha'+\beta'+\gamma'\ldots,$$

$\alpha, \beta, \gamma \ldots \alpha', \beta', \gamma' \ldots$, may be treated as a diptych, and $, b, c \ldots$ $a', b', c' \ldots$ be deduced by equations of the same form as before†: this explains a seeming paradox, that when $y=x^n+cx^{n-2}+\ldots$, so that it contains only $(n-1)$ disposable constants, we can satisfy the equation

$$y^2+y'^2=(x^n+hx^{n-1}+kx^{n-2}+\ldots)^2,$$

which involves the satisfaction of $2n$ equations by $(2n-1)$ unknowns.

(26) It is well worthy of notice that if s is the length of the arc of the compound logarithmic wave $\eta=\Sigma\alpha \log(x+a)+\Sigma\alpha' \log(x+a')$, where the α and α' quantities form a diptych if s be the length of the arc reckoned from a certain point in the curve

$$s-x=\Sigma\alpha \log(x+a)-\Sigma\alpha' \log(x+a').$$

Such a curve is called a rectifiable compound logarithmic wave; and if the a and a' series and the α, $-\alpha'$ series agree term for term, it is said to be symmetrical.

(27) From what has been stated in article (18), it follows that the total number of reducible cyclodes of the order $2n$ is the same as the total number of rectifiable compound logarithmic waves that can be formed with $2n$ non-coincident wavelets of any two given species; and this remarkable equality also subsists when only the symmetrical cyclodes and the symmetrical waves form the subject of comparison. If the class $2m$ be given as well as the order $2n$ of the cyclodes, both the above equalities continue to apply when $2m$ of the wavelets are limited to be of one, and $2n-2m$ of the other species.

(28) The diptychal equations may be applied to the solution of the diophantine equation $y^2+y'^2=\square$ when y is any rational function of x of which the numerator and denominator are each of a given order.

The distinction of class, genus, and species will be still applicable as before, but there will be a fall in the number of solutions when the degree of

* Thus by substitution and multiplication we obtain the important equality

$$\frac{\zeta(abc\ldots)}{\zeta(a'b'c'\ldots)}=\pm\sqrt{\frac{\alpha'\beta'\gamma'\ldots}{\alpha\beta\gamma}},$$

from which it follows that, although the *maximum denumerant* of a diptych of the class μ is $\Pi(\mu-1)\Pi\mu$ (see art. 9), the degree of the resolving equation need never exceed $\frac{\Pi(\mu-1)\Pi(\mu)}{2}$. [$\zeta$ I use here, as of old elsewhere, to denote the product of the differences of the quantities which it precedes taken in some certain prescribed order of succession.]

† I mean that the equation-system above written may be *transformed* into the equation-system of article (8).

y in x is zero, *i. e.* when the given orders of numerator and denominator are the same, so that the rational fraction of the degree is zero: apart from this cause of difference, the number of solutions corresponding to any given diptych will be governed in all cases solely by the principle of parallel equalities.

(29) It is obvious that the theorems giving the value of the D and Δ^* denumerants for any given class, combined with the theorems giving their sum for any given given class and order, amount to independent theorems in the theory of partitions; and as regards the Δ denumerants these theorems belong to the subject of partitions of a comparatively simple character. For instance, the table of Δ for the fourth class enables us to affirm that understanding x, y, z, t to be all distinct quantities, and meaning by $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$ the denumerants (*i. e.* the number of positive-integer solutions, *zeros* excluded) of the indeterminate equations or equation pairs below written,

$$x+y+z+t=n\,;\quad x+y+2z=n\,;\quad x+y=\frac{n}{2}\,;\quad x+3y=n\,;$$

$$\left[\begin{array}{l} x+y+z+t=n \\ x+y=z+t \end{array}\right];\qquad \left[\begin{array}{l} x+y+2z=n \\ x+y=2z \end{array}\right];$$

we have

$$9(\delta_1-\delta_2)+4(\delta_2-\delta_6)+\delta_3+\delta_4+7\delta_5+3\delta_6=\frac{(k+1)^2k}{2} \text{ when } n=2k,$$

$$\text{or } =\frac{(k+1)k^2}{2} \text{ when } n=2k-1\dagger.$$

Moreover, throughout the whole theory, as, *ex. gr.*, in certain marvellous formulæ of reduction which I have obtained concerning the maximum Δ denumerants, say of the diptych $[a^\alpha.b^\beta.c^\gamma....]^2$, problems of partition of numbers occur and recur again and again, some of a known, others of a totally new kind. In this particular we see another point of resemblance between the theory of reducible cyclodes and that of geometric transformations.

(30) Annexed is a Table of the genera of diptychs ‡ (the vacuous one not

* I shall henceforth call D the total and Δ the special denumerant, the Δ^* referring to solutions of either kind indifferently, the *special* to symmetrical solutions exclusively.

† The left-hand side of this equation, *i. e.* $9\delta_1+4\delta_2+\delta_3+\delta_4-2\delta_5-\delta_6$ it will easily be seen may be expressed in terms of $\underset{1:1:1:1}{\overset{n-4}{\cdot}}, \underset{1:1:2}{\overset{n-4}{\cdot}}, \underset{2:2}{\overset{n-4}{\cdot}}, \underset{1:3}{\overset{n-4}{\cdot}}, \underset{4}{\overset{n-4}{\cdot}}$, where $\underset{p:q:r\,\&c.}{\overset{n}{\cdot}}$ means the number of ways of making up the number n with p, q, r, &c.; but the expression will not be a linear function of these quantities, inasmuch as δ_1, δ_2 introduce combinations of the second order. The term (-4) in the symbolical numerators arises from the fact of zero values of x, y, z, t being inadmissible.

‡ If we choose to regard the union of two molecules (alæ), each of like weight (amount) and containing a like number of atoms (elements), as a chemical duad, we see that it is possible for such a duad to be resolved in one or more, and sometimes in a great variety of ways, into two or more subordinate duads, satisfying the same law of number and

excepted) of the fourth class. I give it *subject to correction*. The calculations require great care to exclude repetitions or incompatible equalities and to guard against omissions.

It will be recollected that the sum of the terms is always the same for each ala; so that if the alæ agree in three places, they must agree also in the fourth. When there exist binomial parallelisms, one only of the conjugates is exhibited in the Table, and in all cases only such (and in their simplest form) as are sufficient to define the genus. The mononomial parallel equalities are of course represented by the use of the same letter for the equal terms. The binomial parallelisms are indicated by affixing the same mark to all the terms which enter into each side of the equation constituting the expression of such parallelism.

The lines of division in the Table are introduced to give greater facility for exhaustive verification; the artificial groups of genera thus formed are those in which respectively four, two, one, or no element contained (singly or repeatedly) in one ala are or is contained also in the other ala.

Diptychs.

a	b	c	d	a	b	c	d
a	b	c	c	a	b	c	c
a	a	c	c	a	a	c	c
a	a	a	d	a	a	a	d
a	a	a	a	a	a	a	a
$\dot{a}$	$\dot{b}$	c	d	a	b	$\dot{c}$	$\dot{d}$
$\dot{a}$	$\dot{b}$	c	c	a	b	$\dot{c}$	$\dot{c}$
a	b	c	d	α	β	c	d
a	b	c	c	α	β	c	c
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	$\dot{\beta}$	c	d
$\dot{a}$	b	c	$\dot{d}$	α	β	$\dot{c}$	d
a	b	c	d	α	c	c	d
a	b	c	d	c	c	c	d
a	b	c	d	c	d	c	d

Diptychs.

a	b	c	d	α	β	γ	d
a	b	c	d	α	β	d	d
a	b	c	d	d	d	d	d
a	$\dot{b}$	$\dot{c}$	d	$\dot{\alpha}$	β	γ	$\dot{d}$
$\dot{a}$	a	$\dot{c}$	d	$\dot{\alpha}$	β	γ	$\dot{d}$
$\dot{a}$	$\dot{a}$	a	d	$\dot{\alpha}$	β	γ	$\dot{d}$
$\dot{a}$	a	c	$\dot{d}$	$\dot{\alpha}$	α	γ	d
a	b	c	d	α	β	γ	δ
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	β	γ	δ
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	β	α	δ
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	β	α	β
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	$\dot{\alpha}$	α	δ
$\dot{a}$	$\dot{b}$	c	d	$\dot{\alpha}$	$\dot{\alpha}$	α	α
$\dot{a}$	$\dot{b}$	c	c	$\dot{\alpha}$	β	α	δ

weight; and it is really quite within the bounds of possibility that in establishing and determining diptychical genera and species, we are at the same time solving a problem of the Chemistry of the Future. The combination of two similar ("conformable") molecules (alæ) into a duad (diptych) conveys a sort of image of Brodie's idea of the combination of a substance such as oxygen with itself; and in fact the whole theory of diptychical partition and combination forcibly reminds one of that writer's profound morphological views of the nature of chemical being, action, and passion.

Diptychs.

$\dot{a}$	$\dot{b}$	c	c	$\dot{\alpha}$	$\dot{\beta}$	α	β
$\dot{a}$	$\dot{b}$	c	c	$\dot{\alpha}$	$\dot{\alpha}$	α	δ
$\dot{a}$	$\dot{b}$	a	d	$\dot{\alpha}$	$\dot{\beta}$	α	δ
$\dot{a}$	$\dot{b}$	a	d	$\dot{\alpha}$	$\dot{\beta}$	γ	β
$\dot{a}$	$\dot{b}$	a	d	$\dot{\alpha}$	$\dot{\beta}$	α	β
$\dot{a}$	$\dot{b}$	a	d	$\dot{\alpha}$	$\dot{\alpha}$	α	δ
$\dot{a}$	$\dot{b}$	a	b	$\dot{\alpha}$	$\dot{\beta}$	α	β
$\dot{a}$	$\dot{b}$	a	b	$\dot{\alpha}$	$\dot{\alpha}$	α	α

Diptychs.

$\dot{a}$	$\dot{a}$	a	d	$\dot{\alpha}$	$\dot{\delta}$	δ	δ
$\ddot{a}$	$\dot{b}$	$\acute{c}$	d	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\gamma}$	δ
$\ddot{a}$	$\dot{b}$	$\acute{c}$	d	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\alpha}$	δ
$\ddot{a}$	$\dot{b}$	$\acute{c}$	d	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\alpha}$	δ
$\ddot{a}$	$\dot{b}$	$\acute{c}$	c	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\alpha}$	δ
$\ddot{a}$	$\dot{b}$	$\acute{c}$	c	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\alpha}$	β
$\ddot{a}$	$\dot{b}$	$\acute{a}$	d	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\alpha}$	δ
$\ddot{a}$	$\dot{b}$	$\acute{a}$	d	$\ddot{\alpha}$	$\dot{\beta}$	$\acute{\gamma}$	β

In the last seven diptychs it will be observed double binomial parallelisms occur; thus, *ex. gr.*, in the last of all $a+b=\alpha+\beta$, $a+a=\alpha+\gamma$.

The total number above enumerated is 44, but not improbably one or more (most likely an odd number) have been overlooked. The first group contains only one species to each genus; not so for the other groups. Thus, ex. gr., $a\ b\ c\ d \mid \alpha\ \beta\ c\ d$ represents the two species,

$$\begin{array}{cccc|cccc} a & b & c & d & \alpha & \beta & c & d \\ a & b & c & d & \alpha & \alpha & c & d \end{array}$$

So, again, $a\ b\ c\ d \mid a\ \beta\ \gamma\ d$ represents the several species,

a	b	c	d	α	β	γ	d
a	b	c	d	α	β	β	d
a	b	c	d	α	α	α	d
a	a	c	d	α	β	β	d
a	a	c	d	α	α	α	d;

and $a\ b\ c\ d \mid \alpha\beta\ \gamma\ \delta$ will represent, in like manner, the several species,

a	b	c	d	α	β	γ	δ
a	b	c	d	α	β	γ	γ
a	b	c	d	α	β	β	β
a	b	c	d	α	α	α	α
a	b	c	c	α	β	γ	γ
a	b	c	c	α	β	β	β
a	b	c	c	α	α	α	α
a	b	b	b	α	β	β	β
a	b	b	b	α	α	α	α
a	b	c	d	α	α	γ	γ
a	b	c	c	α	α	γ	γ
a	b	b	b	α	α	γ	γ

But $a\ a\ c\ c \mid \alpha\ \alpha\ \gamma\ \gamma$ would belong to a different genus; for in such diptych

we should have $2a+2c=2\alpha+2\gamma$, and consequently $a+c=\alpha+\gamma$ a new parallel; and *à fortiori* $a\,a\,a\,a \mid \alpha\,\alpha\,\gamma\,\gamma$ belongs to a different genus. The algebraical denumerants will be the same for all the last-written twelve species, viz. Π3 . Π4, *i. e.* 144, the maximum value for the fourth class,—the arithmetical denumerants, on the other hand, being respectively 144, 72, 24, 6, 36, 12, 3, 4, 1, 36, 18, 6. Possibly it may turn out that the number of species may be given by as simple or a simpler function of the index of the class as or than that which expresses the number of genera. It seems very desirable to ascertain if either or both is or are amenable to algebraical or analytical quantification.

As Crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of Invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich Spiral has sprung the vast and interminable Calculus of Cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of denumeration, and, by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the Scriptural comparison of the Kingdom of Heaven "to a grain of mustard-seed which a man took and cast into his garden, and it grew and waxed a great tree, and the fowls of the air lodged in the branches of it"*.

(31) The annexed computation of the denumerants of *diptychs* of the 3rd class and 8th and 9th orders respectively is given in illustration of articles (20) and (13).

The triadic partitionments of 8, it will be observed, are 1 1 6 : 1 2 5 : 1 3 4 : 2 2 4 : 2 3 3. Those of 9, 1 1 7 : 1 2 6 : 1 3 5 : 1 4 4 : 2 2 5 : 2 3 4 : 3 3 3. And of 7, 1 1 5 : 1 2 4 : 1 3 3 : 2 2 3.

In the D column the denumerants of the unconformable diptychs are *doubled*, in order to save the necessity of transposing the *alæ* in summing for the total denumerant [the $Q_{r,s}$ of art. 13].

* The three most remarkable and almost simultaneous births of time for the current year are, I think, Janssen's and Lockyer's hydrogenous solar chromosphere, Tyndall's indefinitely attenuated cometary matter, and the still more impalpable and shadowy product of cerebration embodied in diptychs with their *quasi* chemical composition and parallels stretching between and connecting, as it were, with forces of affinity the atomic elements of the associated geminate molecules, in this sketch set forth. All three theories have originated alike in observation, followed up by processes of experiment and verification working with very different instruments, but having the same *primum mobile* in the human intelligence. On second thoughts I ought to tack on to this list of memorabilia, which must for ever make 1869 stand out in the Fasti of science, Capt. Andrew Noble's mechanical invention for measuring up to the millionth part of a second the rate of motion of a shot inside a cannon and Dr. Christian Wiener's wonderful realization in stereoscopic drawings of the Salmon-Cayley 27 lines on a cubic surface on the one hand, and on the other (Hermite's pupil, pupil worthy of his master) M. Camille Jourdan's surprising discovery of their application to the trisection of Abelian functions. Surely with as good reason as had Archimedes to have the cylinder, cone, and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosiheptagram to be engraved on theirs. Spirit of the Universe! whither are we drifting, and when, where, and how is all this to end?

Diptychs.		D.
116	116	1
116	125	8
116	134	8
116	224	6
116	233	6
125	125	7
125	134	20
125	224	8
125	233	10
134	134	7
134	224	10
134	233	8
224	224	1
224	233	4
233	233	1
$\frac{\Pi 7\ \Pi 6}{\Pi 3\ \Pi 2\ \Pi 5\ \Pi 4} =$		105

Diptychs.		D.
115	115	1
115	124	8
115	133	4
115	223	6
124	124	7
124	133	10
124	223	8
133	133	1
133	223	4
223	223	1
$\frac{\Pi 6\ \Pi 5}{\Pi 3\ \Pi 2\ \Pi 4\ \Pi 3} =$		50*

Diptychs.		D.
117	117	1
117	126	8
117	135	8
117	144	4
117	225	6
117	234	12
117	333	2
126	126	7
126	135	20
126	144	10
126	225	8
126	234	20
126	333	4
135	135	7
135	144	10
135	225	10
135	234	20
135	333	2
144	144	1
144	225	6
144	234	8
144	333	2
225	225	1
225	234	8
225	333	2
234	234	7
234	333	2
333	333	0
$\frac{\Pi 8\ \Pi 7}{\Pi 3\ \Pi 2\ \Pi 6\ \Pi 5} =$		196

* This Table has been already given, differently arranged, at foot of page 13, where, however, the numbers corresponding to pairs in the D column are not, as here, doubled.

(32) The Tables following illustrate the values assigned to the special denumerants (Δ column) in the Tables of arts. 21, 22. The diptychs being of the conformable kind, are, for greater simplicity, represented each by a single ala. The asterisk at the side distinguishes those diptychs in which one or more binomial parallel quantities exist.

Class=8. Order=28.	
Ala.	Δ.
1.1.1.11	1
1.1.2.10	4
1.1.3.9	4
1.1.4.8	4
1.1.5.7	4
1.1.6.6	1
1.2.2.9	4
1.2.3.8	9
1.2.4.7	9
*1.2.5.6	7
1.3.3.7	4
*1.3.4.6	7
1.3.5.5	4
1.4.4.5	4
2.2.2.8	1
2.2.3.7	4
2.2.4.6	4
2.2.5.5	1
2.3.3.6	4
*2.3.4.5	7
2.4.4.4	1
3.3.3.5	1
3.3.4.4	1
$\frac{\Pi(5+2-1)^2}{\Pi 1\ \Pi 2\ \Pi 4\ \Pi 5} =$	90

Class=10. Order=30.	
Ala.	Δ.
1.1.1.1.11	1
1.1.1.2.10	6
1.1.1.3.9	6
1.1.1.4.8	6
1.1.1.5.7	6
1.1.1.6.6	2
1.1.2.2.9	9
1.1.2.3.8	21
1.1.2.4.7	21
1.1.2.5.6	19
1.1.3.3.7	9
*1.1.3.4.6	19
1.1.3.5.5	9
1.1.4.4.5	9
1.2.2.2.8	6
*1.2.2.3.7	20
1.2.2.4.6	21
1.2.2.5.5	9
1.2.3.3.6	21
*1.2.3.4.5	39
1.2.4.4.4	6
*1.3.3.3.5	5
1.3.3.4.4	9
2.2.2.2.7	1
2.2.2.3.6	6
2.2.2.4.5	6
2.2.3.3.5	9
2.2.3.4.4	9
*2.3.3.3.4	5
3.3.3.3.3	0
$\frac{\Pi 6.\Pi 7}{\Pi 4.\Pi 5(\Pi 2)^2} =$	315

*Class=10. Order=32.	
Ala.	Δ.
1.1.1.1.12	1
1.1.1.2.11	6
1.1.1.3.10	6
1.1.1.4.9	6
1.1.1.5.8	6
1.1.1.6.7	6
1.1.2.2.10	9
1.1.2.3.9	21
1.1.2.4.8	21
1.1.2.5.7	21
1.1.2.6.6	9
1.1.3.3.8	9
1.1.3.4.7	21
1.1.3.5.6	21
1.1.4.4.6	9
1.1.4.5.5	9
1.2.2.2.9	6
*1.2.2.3.8	20
1.2.2.4.7	21
*1.2.2.5.6	19
1.2.3.3.7	21
*1.2.3.4.6	41
1.2.3.5.5	21
*1.2.4.4.5	19
1.3.3.3.6	6
*1.3.3.4.5	20
1.3.4.4.4	6
2.2.2.2.8	1
2.2.2.3.7	6
2.2.2.4.6	6
2.2.2.5.5	2
2.2.3.3.6	9
*2.2.3.4.5	19
2.2.4.4.4	2
2.3.3.3.5	6
*2.3.3.4.4	8
3.3.3.3.4	1
$\frac{(\Pi 7)^2}{(\Pi 5\ \Pi 2)^2} =$	441

* *i. e.* the cyclode is of such class and order, but the diptych is of class 5 and order 16, and so for the rest. I repeat that the diptychical class and order is the number of elements and their sum in *each* ala, and must be doubled to give the class and order of the corresponding cyclode.

Class = 10. Order = 34.	
Ala.	Δ.
1.1.1.1.13	1
1.1.1.2.12	6
1.1.1.3.11	6
1.1.1.4.10	6
1.1.1.5.9	6
1.1.1.6.8	6
1.1.1.7.7	2
1.1.2.2.11	9
1.1.2.3.10	21
1.1.2.4.9	21
1.1.2.5.8	21
$\dot{1}.1.\dot{2}.\dot{6}.\dot{7}$	19
1.1.3.3.9	9
1.1.3.4.8	21
$\dot{1}.1.\dot{3}.\dot{5}.\dot{7}$	19
1.1.3.6.6	9
$1.\dot{1}.\dot{4}.\dot{4}.\dot{7}$	8
1.1.4.5.6	21
1.1.5.5.5	2
1.2.2.2.10	6
$1.\dot{2}.\dot{2}.\dot{3}.9$	20
1.2.2.4.8	21
1.2.2.5.7	21
1.2.2.6.6	9
1.2.3.3.8	21

Class = 10. Order = 34.	
Ala.	Δ.
$\dot{1}.\dot{2}.\dot{3}.\dot{4}.7$	43
$\ddot{1}.\ddot{2}.\ddot{3}.\ddot{5}.\ddot{6}$	41
$1.\dot{2}.\dot{4}.\dot{4}.\dot{6}$	20
$\dot{1}.\dot{2}.\dot{4}.\dot{5}.5$	19
1.3.3.3.7	6
$\dot{1}.\dot{3}.\dot{3}.\dot{4}.\dot{6}$	19
$\dot{1}.\dot{3}.\dot{3}.\dot{5}.5$	8
$1.\dot{3}.\dot{4}.\dot{4}.\dot{5}$	20
1.4.4.4.4	1
2.2.2.2.9	1
2.2.2.3.8	6
2.2.2.4.7	6
2.2.2.5.6	6
2.2.3.3.7	9
2.2.3.4.6	21
2.2.3.5.5	9
2.2.4.4.5	9
2.3.3.3.6	6
$\ddot{2}.\ddot{3}.\ddot{3}.\ddot{4}.\ddot{5}$	18
2.3.4.4.4	6
3.3.3.3.5	1
3.3.3.4.4	2
$\frac{\Pi 8\ \Pi 7}{\Pi 6\ \Pi 5(\Pi 2)^2} =$	588
Here $s=5, r=12, k=2, h=6$ *.	

In the above Table, and those which follow, instead of using asterisks (danger-signals, so to say) as in the three Tables immediately preceding to indicate the existence of parallel equalities, I have employed the much more perfect method of connotation which actually exhibits such equalities to the eye.

* See second footnote, p. 6.

Class=10. Order=40.	
Ala.	Δ.
111116	1
111215	6
111314	6
111413	6
111512	6
111610	6
111710	6
11189	6
112214	9
112313	21
112412	21
112511	21
112610	21
11279	21
11288	9
113312	9
113411	21
113510	21
11369	21
11378	21
114410	9
11459	21
11468	21
11477	9
11558	9
11567	21
11666	2
122213	6
$\dot{1}\dot{2}\dot{2}\dot{3}12$	20

Class=10. Order=40.	
Ala.	Δ.
122411	21
122510	21
12269	21
$\dot{1}2\dot{2}\dot{7}\dot{8}$	19
123311	21
$\dot{1}\dot{2}\dot{3}\dot{4}10$	43
12359	45
$\dot{1}2\dot{3}\dot{6}\dot{8}$	43
12377	21
12449	21
$\ddot{1}\dot{2}\ddot{4}\ddot{5}8$	41
$\dot{1}\dot{2}4\dot{6}\dot{7}$	43
12557	21
$\dot{1}\dot{2}\dot{5}66$	19
133310	6
13349	21
$\dot{1}\dot{3}\dot{3}\dot{5}8$	20
13367	21
13448	21
$\dot{1}\dot{3}\dot{4}\dot{5}\dot{7}$	43
$\dot{1}\dot{3}\dot{4}66$	19
13556	21
$\dot{1}\dot{4}\dot{4}\dot{4}\dot{7}$	5
14456	21
14555	6
222212	1
222311	6
222410	6
22259	6

Class=10. Order=40.	
Ala.	Δ.
22268	6
22277	2
223310	9
22349	21
22358	21
$2\dot{2}\dot{3}\dot{6}\dot{7}$	19
22448	9
$2\dot{2}\dot{4}\dot{5}\dot{7}$	19
22466	9
22556	9
23339	6
$\dot{2}\dot{3}\dot{3}\dot{4}8$	20
23357	21
23366	9
23447	21
$\ddot{2}\ddot{3}\ddot{4}\ddot{5}\ddot{6}$	39
23555	6
$\dot{2}\dot{4}\dot{4}\dot{4}\dot{6}$	5
24455	9
33338	1
33347	6
33356	6
33446	9
33455	9
$\dot{3}\dot{4}\dot{4}\dot{4}\dot{5}$	5
44444	0
$\left(\frac{\Pi 9}{\Pi 2\ \Pi 7}\right)^2 =$ Here $s=5$, $r=15$, $k=2$, $h=7$.	1296*

* The *fall* in the arithmetical denumerant due to the existence of binomial parallelisms

Any one desirous of mastering the foregoing exposition (the cradle of a theory complementary to the existing theory of algebraical curves) will do well to possess himself of the ideas conveyed by the new or newly applied terms of art scattered up and down through the outline and resumed in the annexed stenograph. Continued Involutes; Cyclodes, their Pole and Parabolic function; Unicursal Relation; Norwich Spiral; Reducible Cyclodes and their Bifid functions; Algebraico-Diophantine condition; Symmetrical, Congeminate and Contrageminate forms; Compound Logarithmic Waves and Wavelets; Partitionments; Representative Diptychs, their Alæ conformable and unconformable; Monistic and Dualistic equation-systems; Algebraical and Arithmetical Denumeration; Zigzag Multiplication of contiguous lines or columns; Pneumatic Analysis; Evaporation (partial and total); Consolidation; Reduction; Parallel Equalities; Mononomial, Binomial, and Polynomial Parallels (Reducible and Irreducible); Connotation; Order, Class, Genus, and Species of Cyclodes and Diptychs; Special and Total Denumerants. The quotation at the head of the paper is from an article by M. Réville in the 'Revue des deux Mondes' for the present July, "sur la Science des Religions." I have since found that a strikingly similar passage occurs in Dr. Mansel's Bampton Lectures:—"The only test *of truth* is *the harmonious consent of all human faculties*."

K House, Woolwich Common, July 12, 1869.

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in the 4th or 5th diptychical class is easily recollected by aid of the following rule. Every such parallelism causes a fall of 2 units unless two elements in the parallel are identical, in which case the fall is of a single unit; but in applying this rule elements of the same name are not to be counted more than once over in the same parallel. Calling the two varieties of binomial conjugation A and B respectively, it will be found that we may have B or A, or A and B, or A and A, or A and A and A coexisting in the same diptych of the 5th class, causing respectively falls of 1, 2, 3, 4, 6 units in the corresponding values of Δ. It may not unlikely be possible to give rules for assigning the effect of any polynomial parallel on the denumerant in all cases; and then the theory will be greatly simplified, as we shall have only to deal with diptychs of the form $a^r b^s c^t \ldots \mid \alpha^\varsigma \beta^\sigma \ldots$ freed from all latent parallelisms. I am virtually in possession of a complete theory of the special denumerants for diptychs of the form $[a^r.\, b^s.\, c^t..]^2$ on the supposition of the absence of binomial or other polynomial parallel equalities. As regards the method of connotation for singling out the terms which enter into any parallel, a remark may be made which is of some importance, viz. that given such terms, and provided that the parallelism is *irreducible*, *i. e.* not implied in parallelisms of a lower order, the resolution of the parallel into an equation can be effected in only one way. Hence, as we may always reject *reducible* parallels, the method of connotation is exempt from all ambiguity in its application to the single *alæ* of conformable diptychs. As to diptychs with the alæ both expressed, the method of connotation actually exhibits not only the parallels, but the equations which they respectively contain.

It is worth while to notice how gradually and slowly the final form of the theory is evolved. The first reducible cyclode is of the 2nd order. The plurality of such cyclodes only begins to show itself in the 6th order; the existence of non-symmetrical ones, in the 8th; congeminate pairs, in the 12th; reduction in the value of Δ (the special denumerant) owing to the existence of parallel equalities (other than identities), in the 16th. In like manner, I discovered in the theory of partitions that it is not until we have entered upon the discussion of ternary equation-systems that we can be said to have cleared out of the narrows and to be sailing on the open sea; for only then do the most essential features of the theory and its geometrical basis begin to disclose themselves. A very similar remark applies to my unpublished generalization of Gauss's method of interpolation, which I have extended to multiple integrals of all degrees of multiplicity. So, too, it is well known that the algebraical or geometrical theory of forms does not exhibit itself in its true colours until we have passed beyond the case of the second degree, which is quite a world within itself *conditionally unlimited*, incongruent as this association of epithets may sound to the unpurged ears of the Hamiltonian school of metaphysicians.

Feb. 11*th*, 1869.

Prof. CAYLEY, President, in the Chair.

Mr. W. M. Ramsay was proposed for election.

Mr. J. J. Walker read a paper

On Tangents to the Cissoid.

1. The principal object of the following short paper is to suggest a method of drawing the three tangents from a given point to cubics of the third class, by determining either the circle passing through the three points of contact, or that passing through the points in which the tangents intersect the cubic again. This can be effected most completely and easily in the case of the Cissoid of Diocles; and the necessary formulæ for this case I proceed to investigate, using either three or two lines of reference, as may be more convenient—either the tangent at the cusp (y), a line through the cusp perpendicular to it (x) and the asymptote (z), or the first two of these lines only; the transition being made by writing $2a-x$ for z, where a is the radius of the generating circle.

2. The equation to the cissoid is well known to be

$$U = x^3 - y^2 z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and if tangents PT_1, PT_2, PT_3 be drawn from any point $P(x'\ y'\ z')$, then (Salmon, "Higher Plane Curves," p. 68)

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right) U = 0, \text{ or}$$

$$3x'x^2 - z'y^2 - 2y'yz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

will be a conic passing through the three points of contact $T_1\ T_2\ T_3$ and through the cusp (O).

To deduce from (2) the equation to the circle $T_1\ T_2\ T_3$, it will be more convenient to use bilinear coordinates x and y. Writing then $2a-x$ for z, (2) becomes

$$3x'x^2 - z'y^2 + 2y'xy - 4ay'y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying this by $\frac{x}{y}$ and substituting, for $\frac{x^3}{y}$, $(2a-x)y$, and, for z', $(2a-x')$,

$$2y'x^2 - 2(x'+a)xy - 4ay'x + 6ax'y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Multiplying (3) by $\frac{6x'}{x}$, (4) by $\frac{4y'}{x}$, adding and substituting again from (1) x^2+y^2 for $\frac{2ay^2}{x}$, there results

$$3x'z'(x^2+y^2) - 2a(9x'^2 + 4y'^2)x + 4ay'z'y + 16a^2y'^2 = 0 \ldots\ldots\ldots\ldots (5),$$

which is the equation to the circle passing through $T_1\ T_2\ T_3$, the points of contact of tangents to the Cissoid drawn from $P(x'y')$.

3. In terms of the trilinear coordinates xyz, the equation to the same circle is

$$3x'(z'-3x')x^2+3x'z'y^2+4y'^2z^2+2y'z'yz+(4y'^2-9x'^2)zx+2y'z'xy=0 \quad \ldots\ldots\ldots(6).$$

Besides the three points of contact T_1, T_2, T_3, in which (in general) this circle meets the Cissoid, it intersects it in a fourth, extraneous, point (T_0 suppose), which it is necessary to discriminate from the other three. This may be effected as follows.

If (6) be multiplied by y^4, and z be then eliminated by substituting [from (1)] x^3 for y^2z, there results a homogeneous equation of the sixth degree in xy, which is divisible by x^2+y^2, the quotient, viz.

$$3x'z'y^4+2y'z'xy^3-9x'^2x^2y^2+4y'^2x^4$$

breaks up into the factors

$$(3x'y+2y'x)\ (z'y^3-3x'x^2y+2y'x^3).$$

Now if z be eliminated from (2) after multiplication by y and substitution of x^3 for y^2z, the result is precisely the latter of the above two factors. The first factor, equated with zero,

or $$3x'y+2y'x=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

is therefore the line (OT_0) joining the extraneous point with the cusp,

and $$z'y^3-3x'x^2y+2y'x^3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is the system of three lines OT_1, OT_2, OT_3 joining the cusp with the points of contact of tangents drawn from P.

It will be remarked that the discriminant of (8) is (as of course it should be) $x'^3-y'^2z'$.

4. The circle $T_1T_2T_3$ might be constructed from its equation, by rule and compasses. If X, Y represent the distances of its centre from the cusp, measured along the tangent and from that line respectively, and R its radius, then, from (5),

$$\frac{Y}{a}=-\frac{2y'}{3x'},\quad \frac{X}{b}=\frac{3x'}{z'},\quad R^2=X^2+Y^2-Z^2 \quad \ldots\ldots\ldots(9),$$

where b, Z are lines determined by the relations

$$ab=a^2+Y^2,\quad z' : 12x' :: Y^2 : Z^2;$$

but a readier and more elegant construction is obtained by considering its intersections (Q′Q″) with the generating circle

$$x^2+y^2-2ax=0,\quad \text{or}\quad y^2-zx=0 \quad \ldots\ldots\ldots\ldots\ldots(10).$$

For this purpose multiply (6) by x^2, and substitute, in each term in which z occurs, y^2 for zx; the result, divided by x^2+y^2, viz.,

$$2y'^2y^2+y'z'xy-3x'(2x'-a)x^2=0 \quad \ldots\ldots\ldots\ldots\ldots(11)$$

is the equation to the two right lines joining the cusp with the points

common to the two circles: *i. e.*, the lines OQ′, OQ″. It is readily verified that (11) breaks up into

$$\left.\begin{array}{ll} (OQ') & 2y'y+3x'x=0 \\ \text{and } (OQ'') & y'y-(2x'-a)x=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

From (7) and (12) the following construction for the three points T_0, Q', Q'' is at once deduced (A being the centre of the generating circle).

Let fall PQ perpendicular on OA, and in PQ take K, so that $KQ = \frac{2}{3}PQ$. Join OK; then lines through O making with OA, on the opposite side to P, angles equal and complementary to KOA respectively will meet, one the Cissoid in T_0, the other, the generating circle in Q′. Again, on AO take AL = OQ; join LP, and draw through O a line making with OA an angle equal to QPL (on the same side as P, or on the opposite side, according as OQ is greater or less than one-half of OA), which will meet the generating circle in Q″.

Through the three points T_0, Q, Q′, thus determined, the circle may be described, which by its intersections with the Cissoid will give the points of contact, T_1, T_2, T_3, of tangents drawn from P.

5. The condition for the circle $T_1T_2T_3$ touching the generating circle is that the discriminant of (11), viz.,

$$(7x-2a)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

should vanish. Hence* it appears that if P be taken anywhere on a line perpendicular to the cuspidal tangent at a distance from the cusp equal to one-seventh of the diameter of the generating circle, the circle passing through the points of contact of tangents to the Cissoid drawn from P will have the generating circle as its envelope.

Two other special positions of the point P may be briefly noticed: (*a*) when P is taken on the Cissoid itself, the equation (8) has two equal roots, or a square factor; and the third factor, viz. (after multiplication of (8) by y'^2 and substitution of x'^3 for y'^2z'),

$$x'y+2y'x=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

is the line joining the cusp with the point of contact of the tangent drawn *from* P. To construct this point it is therefore only necessary to produce the ordinate PQ to K′ so that QK′ = 2PQ; the line joining O and K′ will meet the Cissoid in the required point (T_3).

The point T_0 being constructed as above, the tangent at P to the circle described through T_0T_3P will be the tangent at P to the Cissoid also. A yet simpler construction for this tangent may be obtained by constructing the point in which it again meets the Cissoid;—thus: join

* Or by forming the condition that the radical axis of the two circles, viz., $3x'(3x'-z')x-2y'z'y-4y'^2z=0$, should touch $y^2-zx=0$.

the cusp with the point K taken in PQ produced so that QK = ½ PQ [see (17) below]. In other words, draw through O the line harmonic conjugate to OP with respect to the cuspidal tangent and OK.

(4.) When the point P is taken on the asymptote, or $z = 0$, equation (8) reduces to $x^3(3x'y - 2y'x) = 0$ .....(15);

so that the point of contact of the tangent which can be drawn from P to the Cissoid is constructed by producing the line OK of § 3.

6. The construction of the tangents which can be drawn from any point to the Cissoid might have been otherwise effected by considering the circle passing through the three points in which these tangents again meet the curve, and without entering into the details of the proofs, I will sketch the method of procedure which appears to me to give the required results most readily, and state the formulæ arrived at.

The conic $\Delta^2 - 4\Delta' U = 0$, where Δ stands for

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right) U,$$

and Δ' for $\left(x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'}\right) U$,

U, U' being $x^3 - y^2z$, $x'^3 - y'^2z'$ respectively, passes through the points (l_1, l_2, l_3) in which tangents drawn from (x', y', z') meet U again (Salmon, "Higher Plane Curves," p. 68), and will have a contact of the second order with the Cissoid at the point (1) in which the curve is met by the line (OP) joining the cusp with the point from which the tangents are drawn.

The equation to this conic is

$$8x'(4y'^3z' - x'^3)x^3 + 4x'^3z'y^3 + y'^3z^3 + 4y'(2x'^3 - y'^2z')yz^2 - 6x'^2y'^2zx - 12x'^3y'zxy = 0 \quad \text{.....(16).}$$

The system of six lines joining the cusp with the points in which this conic intersects the Cissoid (i.e. the triple line O^3 and the three distinct lines $O1_1$, $O1_2$, $O1_3$) is

$$(x'y - y'x)^3(4x'y^3 - 3x'^2y^2 - y'^2x^2) = 0 \quad \text{.....(17).}$$

Hence it easily appears that the circle passing through 1_1, 1_2, 1_3 is

$$6x'^2(x'^2 + y'^2) - 4(3x'^2 + y'^2)x - 4x'y'z + 2x^2 = 0 \quad \text{.....(18),}$$

or in trilinear coordinates

$$3x'(4x' - 3x)x^2 + 12x'y'xy^2 - 4x'yzx + (x'^2 - 2x')z^2 - 4x'yzy = 0 \quad \text{.....(19).}$$

This circle meets the cuspidal tangent in a fourth, extraneous, point (4_c), the line joining which with the cusp is

$$3x'y - y'x = 0 \quad \text{.....(20).}$$

The lines (Oq', Oq'') joining the cusp with the points in which the same circle meets the generating circle are given by

$$y''^2y^2 - 4x'y'xy + 3x'(4x' - 3x')x^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(21),$$

or
$$(y'y - 3x'x)\{y'y - (4x' - 3x')x\} = 0 \quad \ldots\ldots\ldots\ldots\ldots(22),$$

the discriminant of which is $x'y''^2(25x' - 24a)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$.

The radical axis of the circle $t_1t_2t_3$ and the generating circle is

$$3x'(3x' - 4z')x + 4y'z'y - y''^2z = 0 \quad \ldots\ldots\ldots\ldots\ldots(24).$$

The radical axis of the circles $T_1T_2T_3$ and $t_1t_2t_3$ is

$$9x''^2x - 4y'z'y - 5y''^2z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(25).$$

The radical centre of the above three circles is

$$\left.\begin{aligned} x'(5z' - 3x')x &= 2y'z'y \\ (3x' - 2z')x &= y''^2z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(26).$$

7. It will be remarked that one of the lines Oq', Oq'' and the line Ot_0 make with the cuspidal tangent, on the same side, angles which are together equal to a right angle; as appears from comparing (20) and (22). From the comparison of (7) and (12), it appears that OT_0 and OQ' are similarly related;* also, from comparing (7) and (20), that the tangent of the angle which the line OT_0 makes with the cuspidal tangent to the Cissoid is always double the tangent of the inclination of Ot_0 to the same line, on the other side.

The conic $\Delta'^2 - 4\Delta'U'$ will be an ellipse, parabola, or hyperbola, according as $(x'^3 - y'^2z')\{4y'^2(x'^3 - y'^2z') + 3(x'^2 + y'^2)z'^2\}$ is negative, zero, or positive. Consequently, when P is taken on the Cissoid, it will be a parabola, having contact of the *third* order with the Cissoid at P, and a single contact *at another point, viz., where the tangent at* P meets the curve again.

In the course of a discussion upon the paper, Mr. S. Roberts drew attention to Chasles' extension of the method of Des Cartes for describing tangents to roulettes.

Mr. Clerk-Maxwell next drew attention to J. B. Listing's paper in the 10th vol. of the Göttingen Transactions, on the kinds of Cyclosis in lines, surfaces, and regions of space. If p points are joined into a system by l lines, then since $p-1$ lines are sufficient for this purpose, the remaining $K = l - p + 1$ lines give K independent closed paths. Any other closed path must be compounded of these. If we call s the distance travelled by a point along any path, and

* It seems singular that Ot_0 and OT_0 should have this special relation to one of the pairs $Oq'Oq''$, $OQ'OQ''$ respectively only.

$$L = \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} \right) ds,$$

the line-integral of the quantity, whose components are X, Y, Z, along the path, then if the line-integrals round each of the k cycles are $k_1 \ldots k_k$, the value of L from any one point to any other is

$$L = L_0 + n_1 k_1 + n_2 k_2 + \ldots\ldots + n_k k_k,$$

where $n_1, n_2 \ldots\ldots n_k$ are integral numbers.

As an instance; if

$$X = \frac{dw}{dx}, \quad Y = \frac{dw}{dy}, \quad Z = \frac{dw}{dz},$$

where w is the solid angle subtended at the point xyz by a closed curve, then if one of the cycles of the curve along which the line-integral is taken is enlinked with this closed curve, the corresponding value of k is 4π.

This will be the case if $L = \iint \frac{u}{r^3}\, ds\, ds'$,

where $$\frac{u^2}{r^2} = \left[1 - \left(\frac{dr}{ds}\right)^2\right]\left[1 - \left(\frac{dr}{ds'}\right)^2\right] - \left(r\frac{d^2r}{ds\,ds'}\right)^2,$$

r being the distance from a point on the closed curve s to a point on the closed curve s', and the integral is taken round both curves. This integral is always $4\pi n$, and is a criterion of the curves being linked together or not, depending only on the relations of r, s, and s'.

Prof. Hirst made some further remarks upon the subject which he had brought before the Society at its January meeting. Both communications are included in the following paper:—

On the Degenerate Forms of Conics. *By* T. Archer Hirst, F.R.S.

1. In his memoir "On Curves which satisfy given Conditions," published in the "Philosophical Transactions" for 1868, Professor Cayley has supplemented his very brief contribution to the discussion "On Special Forms of Conics," which appeared some time ago in the pages of the Quarterly Journal (vol. 8), and Messenger (vol. 4) of Mathematics, by describing with precision, and under distinctive names, the three degenerate forms of conics which play so important a part in the recent researches of Chasles, Zeuthen, De Jonquières, Cremona, and others.

In doing so, he has moreover raised, incidentally, this interesting question,—To what extent must the precise forms of these degenerate conics be postulated?

2. With a view of elucidating this question, and, at the same time, of arriving at a correct conception of the points and tangents of a conic which is *on the point* of degenerating, or, in other words, of losing its class or order, I propose, in the present Note, once more to apply the

principle of continuity to the ordinary mode of generating a proper conic by means of homographic pencils of rays, and rows of points.

I say 'once more,' because the subject has, to some extent, been already treated in this manner in several works on Modern Pure Geometry. No author, however, appears to have given to the discussion that prominence to which it is now justly entitled, by virtue of the important researches above alluded to.

3. It is well known that every proper conic may be generated in each of two ways; 1° as the *locus* of the intersection of corresponding rays in two homographic pencils, and 2° as the *envelope* of the connector of corresponding points in two homographic rows.

Generated as a locus, the conic is always touched at the centres of the two pencils by the rays whose correspondents coincide with the connector of those centres.

Generated as an envelope, the conic always touches the axes of the two rows at the points whose correspondents coincide with the intersection of those axes.

The persistence of the two last properties at the moment the conic degenerates, constitutes an essential feature of the following enquiry.

4. The degeneration of a conic thus generated is due either to the relative positions of the generating pencils or rows, or to the special character of their homographic correspondence.

The special kind of correspondence here alluded to, requires a word of explanation. It will be remembered that two primary geometrical forms (*e. g.* two pencils of rays, or two rows of points, or a pencil and a row), are said to be homographic when to each element of one form corresponds one and only one element of the other, and *vice versâ*. Now, ordinarily, no two distinct elements have coincident correspondents; in the special case referred to, however, the correspondents of every element, except one, of each form are coincident, and the correspondent of that excepted one is indeterminate.

The elements of two special homographic forms, whose correspondents are indeterminate, will, for distinction, be termed the *exceptional* elements.*

5. Now, the correspondence being special, two homographic pencils $[p]$ and $[p']$ will, in general, generate a *line-pair* (A, B′), where A and B′ are the exceptional rays of the pencils. For every ray of $[p']$ will

* The simplest instance of the passage from ordinary to special homographic correspondence presents itself when the centre p of a pencil $[p]$, whose rays project the points of a row $[P]$, falls on the axis P of that row. For then the rays which correspond to every point of $[P]$, p excepted, coincide with P, whilst the ray corresponding to the excepted point p is wholly indeterminate; and, conversely, the points corresponding to every ray of $[p]$, P excepted, clearly coincide in p, whilst the correspondent of the excepted ray P is an indeterminate point of the row $[P]$. The general criterion of special homographic correspondence is given in Note A.

intersect its correspondent on A, and every ray of $[p]$ will intersect its correspondent on B′. The point (AB′) will be a double point of this degenerate conic, since with it will coincide the intersection of A and A′ as well as that of B and B′.* The centres p and p', however, will be ordinary points, the tangents at which will be represented by A and B′, respectively, these being the rays of the pencils $[p]$ and $[p']$ which correspond to the connector pp' regarded, successively, as a ray of $[p']$ and of $[p]$. (3).

6. If, without altering the character of the correspondence, the exceptional rays A and B′ be brought to coincidence, by turning the pencils $[p]$ and $[p']$ around their fixed centres, a remarkable change will take place in the nature of the generated conic. Its double point will become indeterminate, and the centres p and p' will simultaneously acquire the character of vertices or terminations. In fact, every ray of the pencil $[p]$, A excepted, will now intersect its correspondent in p, and every ray of $[p']$, with the exception of B′, will cut its correspondent in p'; moreover any point whatever of the coincident rays A and B′ may be now regarded as the intersection of A with A′ and of B with B′. Lastly, in virtue of the property cited in Art. 3, every ray through p or p' is now entitled to rank as a tangent, since its correspondent coincides with the connector $[pp']$; this connector, indeed, has a double claim to be so regarded.

7. The degenerate conic into which the line-pair has been thus transformed, is obviously the *conique infiniment aplatie* of Chasles, the *point-pair* of Prof. Cayley. Its points may be said to constitute a *double* row terminated by p and p', and its tangents to fill the two pencils $[p]$ and $[p']$.

8. A conic generated by two pencils, whose homographic correspondence is of the ordinary type, can only degenerate in two cases, viz.: *first*, when two corresponding rays coincide with the connector of the centres of the pencils, and *secondly*, when these centres are themselves coincident.

9. The first of these cases requires little comment. Every point on the connector is then to be regarded as an intersection of two cor-

* The points (AA′), (BB′) ought not to be regarded as *consecutive* points of the conic, for, on introducing the conception of continuous motion, it will readily be seen that during the rotation of a ray X of the pencil $[p]$, from parallelism with B′ up to coincidence with A, its correspondent X′ in the pencil $[p']$ will remain stationary and coincident with B′; but, before X quits the position A, X′ will make an entire semi-rotation from the position B′, and it is only after this semi-rotation is completed that X can complete its own. The intersection (XX′), therefore, will first describe the portion of B′ between infinity and (AB′), then the portion of A between (AB′) and infinity, next the remaining portion of A, and finally the remaining part of B′. In short, we may regard the double point (AA′), (BB′) as formed by the coincident vertices of a conic which has "sharpened itself to coincide with its asymptotes"—a conception of Prof. Cayley's, which accords well with all that follows.

responding rays; and, by a well known theorem, all other such intersections lie on a right line which, together with the above connector, constitute the degenerate conic (line-pair). It may be observed, however, that the right line in question cannot coincide with the connector, unless the correspondence become special, and in this case, as we have just seen, the conic is transformed from a line-pair to a point-pair.

10. The second of the two cases referred to in Art. 8, merits closer attention. When the centres of two homographic pencils coincide, there are, as is well known, two, and only two, lines A and B′ with each of which two correspondents coincide.* Now since every point of each of these lines may be regarded as an intersection of corresponding rays, the generated conic is obviously a *line-pair* (A, B′). It should be noted, however, not only that all the remaining pairs of corresponding rays intersect in the point (A, B′), but that every line L through this point has a double claim to rank as a tangent of the *line-pair* at that point. For it is a ray of $[p]$ as well as of $[p']$; and to whichever of these pencils it may be considered to belong, its correspondent, in the other, may be regarded as the connector of the centres of both (3).

Each of the coincident centres p and p', in fact, now plays the part of a vertex of the conic, the tangents at which coincide in pairs so as to form an incomplete double pencil terminated by the lines A and B′.

11. When these lines are coincident,† the conic generated by the concentric pencils will be the *line-pair-point* of Prof. Cayley—a right line terminated by coincident points. Every point thereof will represent two coincident, but non-consecutive, points of the conic, and every line through the coincident terminations will represent two non-consecutive tangents.

12. By turning one of the pencils $[p]$ and $[p']$ around the common centre, the lines A and B′, with each of which two corresponding rays coincide, will, after being first real and then coincident, become imaginary. The generated conic, therefore, will transform itself from a real line-pair, through a line-pair-point, to an *imaginary line-pair*,—a geometrical entity to which the name *point-ellipse* has often been given, since all its real points are coincident.

13. An exhaustive consideration of the special cases which present themselves when conics are generated by two homographic pencils having thus revealed the existence of three distinct degenerate forms, it remains to be shown that similar considerations based upon the second method of generation, referred to in Art. 3, lead to precisely the

* See Note C, p. 173, for the conditions under which these double rays will be real, coincident, or imaginary.

† A simple rotation of one of the two pencils $[p]$ and $[p']$ will, in general, suffice to make them so. (Note C.)

same forms, and serve to complete our knowledge of their points and tangents.

The correspondence being of the special kind,* two homographic rows [P] and [P′], of which neither the exceptional points a, b' nor the axes P, P′ are coincident, will generate a point-pair (a, b'); for the correspondent of every point on P′, b' excepted, being at a, and that of every point on P, a excepted, being at b', it is obvious that every connector of a pair of correspondents, that is to say, every tangent of the generated conic, will pass either through a or b', and that two non-consecutive tangents will pass through both, and thus form the double tangent $\overline{aa'}$, $\overline{bb'}$. The axes P and P′ are, of course, ordinary tangents, and a and b', as correspondents of the intersection (PP′), are their points of contact (3).

14. The generated conic changes its character entirely when, without altering the axes or the character of the correspondence, the exceptional points a and b' become coincident. It transforms itself, in fact, from a point-pair (a, b') to a line-pair (P, P′); for, in the first place, an infinite number of connectors of corresponding points now coincide with each axis, that is to say, each axis represents an infinite number of tangents, the points of contact of which, as correspondents of the intersection (PP′), fill both axes; and in the second place, the double tangent $\overline{ab'}$ of the point-pair has become indeterminate in direction, so as to represent any two of the coincident and non-consecutive tangents at the vertices of the line-pair (10).

15. The circumstances under which a conic will degenerate when the homographic correspondence of two rows is of the ordinary kind, are, *first*, the coincidence of two corresponding points in the intersection of the axes P and P′, and *secondly*, the coincidence of the axes themselves.

In the first case, the intersection (PP′) is one of the points of a point-pair; the other being the point in which, as is well known, all connectors of the remaining pairs of correspondents then concur. This latter point, it should be observed, cannot coincide with the first unless the correspondence become special. By such a coincidence, therefore, the point-pair would again become transformed to a line-pair (14).

In the second case, when the axes of the two rows [P] and [P′] are coincident, there will be two, and only two points a and b' in each of which two correspondents are coincident.† Now every line through each of these double points will rank as a tangent of the generated conic, since it connects a pair of correspondents, whilst all other such

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* See Note A.

† As shown in Note B, these double points may be rendered real, coincident, or imaginary, at pleasure.

connectors or tangents will obviously coincide, in pairs, with the coincident axes P and P′. Every point of the latter, in fact, is doubly entitled to be regarded as the point of contact of such a tangent, since it is not only a point of each of the two rows, but it may also be regarded as the correspondent, in each row, of the indeterminate intersection of the axes P and P′ (3).

It is now obvious that the generated conic will be a point-pair when the double points are real, and a line-pair-point when they are coincident; when the double points are imaginary the conic is of course an *imaginary point-pair*, or a double line with no real terminations and no real tangents which do not coincide with the line.

16. To resume, then, we may say that there are but three degenerate forms of proper conics, viz. :—

(i.) The *point-pair* (real or imaginary) whose tangents fill two simple pencils, and whose points constitute a double row terminated by the centres of those pencils.

(ii.) The *line-pair* (real or imaginary) whose points fill two simple rows, and whose tangents constitute a double pencil terminated by the axes of those rows.

(iii.) The *line-pair-point* whose tangents fill a double pencil and whose points form a double row situated on one of the rays of that pencil.

It is scarcely necessary to observe that the double row in (i.) may lie either *between* its extremities or entirely *beyond* them. In the first case the conic may be regarded as a flattened ellipse, in the second as a flattened hyperbola. In like manner the terminated double pencil in (ii.) may fill either of the two vertically opposite spaces into which the terminating lines divide the plane.

Note A.

The homographic correspondence of two rows of points [P] and [P′], as defined in Art. 4, is expressed in the most general manner by the equation

$$xx' + ax + bx' + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where x and x' are the distances of any two corresponding points (correspondents) from fixed origins in the axes P and P′ of the respective rows. If between the arbitrary constants a, b and c the relation

$$c - ab = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

exists, the correspondence will be *special* in the sense indicated in the text; for then, and then only, the relation (1) will assume the form

$$(x+b)\,(x'+a) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

which characterises the specialty by showing that $x = -b$ for all values of x' except one ($x' = -a$), and that for this one x is indeterminate

The values $x = -b$ and $x = -a$, in short, determine the exceptional points of the special rows.

Similar remarks apply, of course, to homographic pencils. The correspondence of two such pencils, in fact, will be special, if that of the rows be so which the pencils determine on any two transversals.

Note B.

Without loss of generality we may suppose the x and x' in equation (1) of the preceding Note A to have a common origin at the intersection of the axes P, P′ of the two homographic rows. Now, to bring these axes into coincidence in the most general manner, we may conceive one of the rows, say P′, to be first slided along its axis, and then that axis turned to the *right* or *left* around the intersection (PP′). The result of these two operations is clearly equivalent to substituting in (1) $a \pm x'$ for x', and then assuming that positive values of x and x' are measured from the same origin in the same direction. Putting $e = \mp 1$, this substitution gives

$$ex x' + (a+a)x + eb x' + ab + e = 0 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

The points (*double-points*) on the common axis in each of which two corresponding points coincide are determined, therefore, by the quadratic

$$e\xi^2 + (a + a + eb)\xi + ab + e = 0 \quad \ldots\ldots\ldots\ldots\ldots (5),$$

whose discriminant

$$\Delta = (a+a+eb)^2 - 4e(ab+e) = a^2 + 2(a - eb)a + (a + eb)^2 - 4e^2 \quad \ldots (6)$$

is a quadratic function of a having, for its own discriminant,

$$\nabla = (a-eb)^2 - (a+eb)^2 + 4e^2 = 4e(e - ab) \quad \ldots\ldots\ldots (7).$$

Two cases, therefore, naturally present themselves for discussion.

First, if the homographic correspondence be special, then, by Note A, $\nabla = 0$, and, as a consequence, Δ is a perfect square, in other words, *positive* for all values of a except the one $(a = eb - a)$ which causes it to vanish. Accordingly the roots of (5) are always real, and we conclude that

1°. *In whatever manner the axes of two special homographic rows be brought into coincidence, the two double points will always be real, though, possibly, coincident.* It is obvious, indeed, from the very nature of special correspondence, that the exceptional points must become double points when the axes of the rows are coincident.

Secondly, if the homographic correspondence be not special, then $\nabla \neq 0$, not vanish, and we may make it either positive or negative at pleasure since to e we may give either of the values ± 1.

If we make ∇ *negative*, then, as is well known, Δ will always be positive, and never vanish, for real values of a. The roots of (5), there-

fore, will be real and distinct, whatever value a may have, and we conclude that

2°. *It is always possible, when the correspondence is not special, to bring the axes of two homographic rows into coincidence in such a manner that the double points shall not only be real and distinct, but shall remain so, however one row may be subsequently slided over the other.*

If we make ∇ *positive*, then there will be two real values of a, which will render $\Delta = 0$, and consequently the roots of (5) real and equal. Moreover, the values of a which lie between these will render Δ negative, and hence the roots of (5) imaginary, whilst all other values will cause Δ to be positive, and therefore the roots of (5) real and distinct. That is to say:

3°. *It is always possible, when the correspondence is not special, so to bring the axes of two homographic rows into coincidence that, by a subsequent sliding of one row over the other, the double points shall be rendered real, coincident, or imaginary, at pleasure.*

Note C.

In rendering two homographic pencils concentric, we have likewise a choice of two ways of bringing their planes to coincidence. It is obvious, therefore, that an investigation similar to the one given in the preceding note would lead to the following three correlative results:

1°. In whatever manner the centres and planes of two special homographic pencils be brought into coincidence, the two double rays will always be real, though possibly coincident.

2°. It is always possible, when the correspondence is not special, so to bring the centres and planes of two homographic pencils into coincidence that the double rays shall not only be real and distinct, but shall remain so, however one pencil may be subsequently turned around its centre in the plane of the other.

3°. It is always possible, when the correspondence is not special, so to bring into coincidence the planes and centres of two homographic pencils that, by a subsequent turning of one around the common centre in the common plane, the double rays shall be rendered real, coincident, or imaginary, at pleasure.

The President drew attention to an important discovery in the Theory of Covariants by Herr Gordan.

The following presents were received during the month:—

"Monatsbericht" for Nov. and Dec., 1868; and

Prof. Cremona's "Sulle Superficie gobbe di quarto grado;" from the author.

March 11th, 1869.

Prof. CAYLEY, President, in the Chair.

Mr. T. Bond Sprague, M.A., was admitted into the Society. Mr. W. M. Ramsay was elected a Member; and Messrs. W. Marsham Adams, B.A., and Mr. J. Walmsley, B.A., were proposed for election.

Prof. Sylvester gave a statement of a "Theorem in the Composition of Logarithmic Waves," and went into details connected with this subject, and its bearing upon Cyclodes and the properties of Numbers.

After some remarks by Mr. Clifford, the President gave an account of a paper he was preparing "On the rational transformation between two spaces.

The following present was made: "Étude sur le déplacement d'une figure de forme invariable, par A. Mannheim;" from the Author.

April 8th, 1869.

Prof. CAYLEY, President, in the Chair.

Messrs. W. Marsham Adams, B.A., and J. Walmsley, B.A., were elected Members; and E. H. Riches, LL.D., was proposed for election. Mr. Adams was subsequently admitted into the Society.

Mr. Archibald Smith read a paper "On the Analytical Formulæ which express the deviations of the Compass." Captain Evans bore testimony to the value of Mr. Smith's investigations. Mr. J. J. Walker next made a short communication, the objects of which were to give the formulæ for the Anharmonic Ratios of the four points of intersection, with respect to either, of a pair of conics in terms of their invariants; also of the points in which a tangent to either meets the four common tangents; to draw attention to the conditions of osculation of the two conics in terms of the invariants; and to suggest that the conditions of double contact, or of a single contact of the third order, between the pair, were not expressible in terms of those invariants.

A discussion ensued, in which the President, Prof. H. J. Smith, and Mr. Roberts took part.

Prof. Smith then gave an account of some recent discoveries he had made in the "Properties of Homographic Figures."

The meeting closed with an explanation by Mr. Clifford of a diagram he had drawn out illustrating applications of Quadric Transformation.

The following presents were received during the month:—

"Mechanical Geometry," by A. H. L. S. Béchaux, B.A.; from the Author.

"Criticism on Mr. Wilson's paper 'On Euclid as a text-book of Elementary Geometry,' by Professors Brioschi and Cremona"; from the Authors.

"Monatsbericht Januar, 1869."

"Crelle's Journal," 69 Band, iv^{tes} Heft.

"Annali di Mathematica."

May 13th, 1869.

Professor CAYLEY, President, in the Chair.

Dr. E. H. Riches was elected a Member. Prof. Newton, of Yale College, was amongst the company present.

Two communications from Mr. C. W. Merrifield, F.R.S., were read by Mr. Tucker, Hon. Sec. Of the former, the following is an abstract:—

On a Geometrical proposition, indicating that the property of the radical axis was probably discovered by the Arabs. By C. W. Merrifield, F.R.S.

The main proposition is among the lemmas which the Arab writers attribute to Archimedes; and the two scholia, which the Arab commentators have added, appear to indicate a knowledge of the radical axis, and of the connexion between the two cases in which the circles do, and do not, really intersect one another.

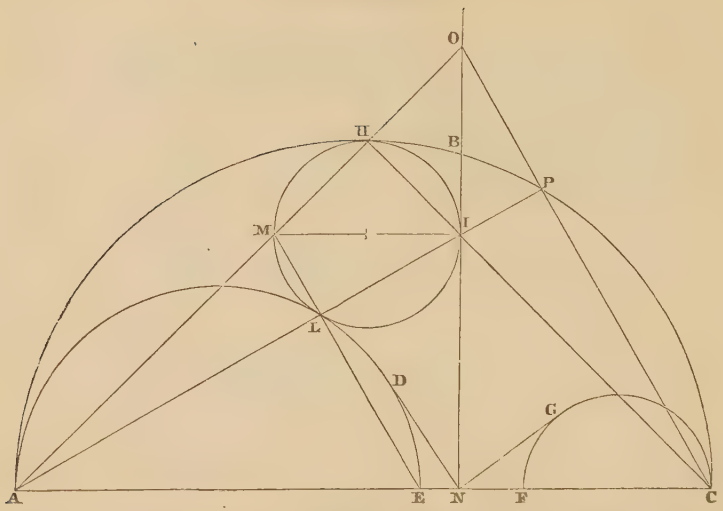
The proposition and scholia are as follows:—

Propositio V. Si fuerit semicirculus AB, et signatum fuerit in ejus diametro punctum C ubicumque, et fiant super diametrum duo semicirculi AC, CB, et educatur ex C perpendicularis CD super AB, et describantur ad utrasque partes duo circuli tangentes illam et tangentes semicirculos, ubique illi duo circuli sunt equales.

Scholium primum Alkantri. Si non fuerint duo semicirculi tangentes, sed mutuo se secantes, et perpendicularis fuerit in loco mutuæ sectionis, idem sequitur.

Scholium secundum Alkantri. Porro secunda est hæc: Dicit, quod si duo semicirculi non sint tangentes, nec se mutuo secantes, sed separati, et perpendicularis transeat per concursum duarum linearum tangentium eos, quæ sint æquales, idem sequitur.

Demonstration.



Let the semicircles ABC, ADE, FGC be as premised, and let a pair of equal tangents NG, ND, touching the circles FGC and ADE in G and D respectively, meet in N on the line AC. And through N let there be drawn NB perpendicular to AC. Let this be touched in I by the circle MLI, which also touches the circle ABC in H and the circle ADE in L.

Through I draw a diameter IM to the circle MLI. This will be parallel to AC, since both are perpendicular to BN. Join AL, LI and EL, LM; then, since L is the point of contact of the two circles MLI and ALE, and AE, MI are parallel diameters, ALI and ELM are right lines. Next, join AM, MH, and CI, IH; then, by the same reasoning, AMH and CIH are right lines. Produce AMH to meet NIB in O, and join OC. Then, in the triangle AOC, ON is perpendicular to AC and CH to AO, and they meet in I. Therefore AI is perpendicular to OC, and meets it in the point P where OC cuts the semicircle ABC.

Moreover, since the angles ALE and APC are right angles, MLE is parallel to OPC, and we have already seen that MI is parallel to AC. Therefore, by similar triangles,

$$AN : MI :: AO : OM :: AC : CE.$$

Therefore the rectangle between AC and MI is equal to the rectangle between AN and CE. Similarly the rectangle between AC and the diameter of the other circle drawn on the other side of BN may be shown to be equal to the rectangle between AF and CN. But $AN \cdot NE = DN^2 = NG^2 = CN \cdot NF$. Add to each side of the equality the rectangle $AN \cdot CN$, and we obtain $AN \cdot CE = CN \cdot AF$. Whence

the rectangle between AC and the diameter of either circle is the same, whence follows the equality of the two circles. Q. E. D.

The date of the Arab writer appears to be as nearly as possible A.D. 1000. The proposition is to be found in the following works, all of which are rare.

(1.) Archimedis Lemmata, London, 1659.

(2.) The Florentine edition of the 5th, 6th, and 7th books of the Conics of Apollonius of Perga, to which the Lemmas of Archimedes are appended.

(3.) Barrow's Archimedes (London, 1675).

The proposition is one of the properties of the ΑΡΒΗΛΟΣ, but differs from all those which have been preserved by Pappus.

The latter brief communication stated a property of spherical triangles, and is as follows :—

If ABC is a spherical triangle, and A′, B′, C′ opposite points to A, B, C on the sphere, so that ACA′, &c., are half great circles, and ABCD a small circle circumscribed to ABC, then the angles which this circle makes with the three sides are severally half the spherical excesses of the triangles A′BC, AB′C, ABC′.

Mr. T. Cotterill then gave an account of his paper on "An Envelope in the Cubic Correspondence of Points." [This paper will be given below.]

Professor Smith and the President made a few remarks on the paper.

Dr. Henrici next stated some of the results aimed at and obtained in his paper

On Series of Curves, especially on the Singularities of their Envelopes; with Applications to Polar Curves. *By* Olaus Henrici, Ph.D.

1. The present paper is based upon a theorem about Discriminants of Discriminants, which I have given in a former paper (Nov. 12th, 1868), and which I repeat here in a more perfect form.

Lemma I.—Let $F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w})$ represent a rational integral function of order p in u, of order q in v, and of order r in w. Let Ω be the discriminant of this function with regard to u, and D the discriminant of this discriminant Ω with regard to v; *then* D *breaks up into three factors*, viz. $D = A \cdot B^2 \cdot C^3$, which are rational integral functions of the coefficients of u, v, and consequently functions of w.

$D = 0$ determines w in such a way that Ω has a double root v; and to these values of v, w correspond,—for $A = 0$, one double root u of F,

but for this root $\frac{\delta F}{\delta v}$ vanishes, or A is the discriminant of F with regard to u, v; for $B=0$, two distinct double roots u of F; for $C=0$, one treble root u of F.

For the orders of these factors I distinguish the following cases:

1°. Every coefficient of u, v is of order r in w, or

$$F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w}) = (u, 1)^p (v, 1)^q (w, 1)^r.$$

It is as follows from my former paper:

Ω	of order	$2q(p-1)$ in v, and $2r(p-1)$ in w,
A	„	$2r[3pq-2(p+q)+2]$ in w,
B	„	$4rq(p-2)(p-3)$ „
C	„	$6rq(p-2)$ „

2°. Let $r=q$, and, writing†

$$F(\overset{p}{u}, \overset{q}{v}, \overset{q}{w}) = (F_0, F_1, \dots F_p \mathbin{)\!(} u, 1)^p,$$

$$F_k = (a_0^{(k)}, a_1, \dots a_q^{(k)} \mathbin{)\!(} v, 1)^q,$$

let either $\quad a_i^{(k)} = (* \mathbin{)\!(} w, 1)^i, \quad$ or $\quad a_i^{(k)} = (* \mathbin{)\!(} w, 1)^{q-i},$

---

† The notation used is somewhat different from that usually adopted.

By $F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w})$ I denote a rational integral function of order p in u, of order q in v, and of order r in w, so that

$$F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w}) = (u, 1)^p (v, 1)^q (w, 1)^r.$$

Or if we write $\quad F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w}) = (F_0, F_1, \dots F_p \mathbin{)\!(} u, 1)^p,$

then $\quad F_k = (a_0^{(k)}, a_1^{(k)}, \dots a_q^{(k)} \mathbin{)\!(} v, 1)^q, \quad$ and $\quad a_i^{(k)} = (* \mathbin{)\!(} w, 1)^r.$

Here every coefficient $a_i^{(k)}$ is of order r in w.

The quantics which occur most frequently in this paper are of the more special form considered in case 2°, where $r=q$, and where either

$$a_i^{(k)} = (* \mathbin{)\!(} w, 1)^i \quad \text{or} \quad a_i^{(k)} = (* \mathbin{)\!(} w, 1)^{q-i},$$

so that the orders in w of $a_0^{(k)}, a_1^{(k)}, \dots a_{q-1}^{(k)}, a_q^{(k)}$ are in arithmetical progression $0, 1, \dots q-1, q$, or $q, q-1, \dots 1, 0$. Such a quantic I denote by $F(\overset{p}{u}; \overset{\overline{q}}{v, w})$, where the double line above the variables v, w means that the orders with respect to w of coefficients of v are in arithmetical progression, as stated. A single line above some variables will express that the quantic is homogeneous in these variables. Thus

$$f(\overset{\overline{p}}{x_1, x_2, x_3}; \overset{r}{w}) = (x_1, x_2, x_3)^p (w, 1)^r.$$

A quantic $F(\overset{p}{u}; \overset{\overline{q}}{v, w})$ results, for instance, from $f(\overset{p}{u}; \overset{\overline{q}}{x_1, x_2, x_3})$ by substituting

so that the orders with respect to w of $a_0^{(k)}, a_1^{(k)} \ldots a_{q-1}^{(k)}, a_q^{(k)}$ are in arithmetical progression $0, 1, \ldots q-1, q$, or $q, q-1, \ldots 1, 0$. This form I shall denote by $\mathrm{F}(\overset{p}{u}; \overset{q}{v, w})$. Then the orders of A, B, C follow from those in case 1°, by substituting $\frac{1}{2}q$ for r. This givesf or B and C the same orders as Dr. Salmon and Mr. Roberts have found for the conditions of two double roots or one treble root of $\mathrm{F}(u)$.

3°. Let $f(\overset{p}{x_1, x_2, x_3}; \overset{r}{w}) = 0$ be the equation to a curve of order p, containing a parameter w. Let us introduce, instead of x_1, x_2, x_3, the parameter v of a pencil of lines, and a quantity u, which determines a point on this line v so that to each value of u belongs one point on the line v and to each point on this line one value of u. The resulting equation will be of the form $\mathrm{F}(\overset{p}{u, v}; \overset{r}{w}) = 0$.

Then A breaks up into two factors

$$\mathrm{A} = f(\overset{p}{\xi_1, \xi_2, \xi_3}; \overset{r}{w}) \cdot \Delta,$$

where ξ_1, ξ_2, ξ_3 are the coordinates of the centre of the pencil, and where Δ is the discriminant of $f(x_1, x_2, x_3)$.* The factor Δ of

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for the coordinates x_1, x_2, x_3 of a point in a plane two variables v, w, of which one fixes a line in a pencil, whilst the other determines a point on this line.

To show that the two cases included in the form $\mathrm{F}(\overset{p}{u}; \overset{q}{v, w})$ are equivalent, let us start with a quantic

$$\mathrm{F} = (u_1, 1)^p (v_1, v_2)^q = (\mathrm{F}_0, \mathrm{F}_1, \ldots \mathrm{F}_p \mathbin{)\!(} u, 1)^p,$$

where $\mathrm{F}_k = (b_0^{(k)}, b_1^{(k)}, \ldots b_q^{(k)} \mathbin{)\!(} v_1, v_2)^q$, and $b_i^{(k)} = (* \mathbin{)\!(} w, 1)^i$.

Making either $v_1 = v$, $v_2 = 1$, or $v_1 = 1$, $v_2 = v$, we obtain two different forms for F, viz., $(* \mathbin{)\!(} u, 1)^p (v, 1)^q$ and $(* \mathbin{)\!(} u, 1)^p (1, v)^q$.

The discriminants of these quantics with regard to u are of the forms

$$(* \mathbin{)\!(} v, 1)^{q(p-1)} \quad \text{and} \quad (* \mathbin{)\!(} 1, v)^{q(p-1)}.$$

Their discriminants with regard to v are identical, because they are both equal to the discriminant of $(* \mathbin{)\!(} v_1, v_2)^{q(p-1)}$. Now, denoting in both cases the coefficient of v^i in F_k by $a_i^{(k)}$, so that $\mathrm{F}_k = (a_0^{(k)}, a_1^{(k)} \ldots \mathbin{)\!(} v, 1)^k$, we have either

$$a_i^{(k)} = b_i^{(k)} = (* \mathbin{)\!(} w, 1)^i \quad \text{or} \quad a_i^{(k)} = b_{q-i}^{(k)} = (* \mathbin{)\!(} w, 1)^{q-i},$$

as in the text.

* This is most easily seen for Cartesian coordinates. Make $x_3 = 1$, and substitute $x_2 = vx_1 + \xi_2$, $x_1 = u$. Then $x_2 = vx_1 + \xi_2$ represents a pencil with parameter v, and u determines a point on the line v. A is the discriminant of $\mathrm{F}(\overset{p}{u, v}; \overset{r}{w})$, or in the present case of $f(\overset{p}{u, vx_1 + \xi_2, 1}; \overset{r}{w})$, with regard to u, v. The actual formation of this discriminant gives the form of A.

$D = f(\xi_1, \xi_2, \xi_3)$ Δ B^2 C^3 is therefore independent of the coordinates of the centre of the pencil.

It is Ω of order $p(p-1)$ in $u+v$ $\delta(p-1)$ in w,

$f(\xi_1, \xi_2, \xi_3)$	"	r	in w:
Δ	"	$3r(p-2)^2$	"
B	"	$2q r(p-2)(p-3)$	"
C	"	$3q r(p-2)$	"

If we form, in any one of these cases, the determinant $\Gamma = \pm \Delta^{\alpha} B^{\beta} C^{\gamma}$ of Ω with regard to w, then Γ becomes a function of u, and B, C are of the same order in u as B, C in w; to each root w of B or C corresponds one root u of B or C respectively, and conversely.

2. Lemma II.—If $F(\overset{p}{x_1}, \overset{p}{x_2}, \overset{p}{x_3}; \overset{r}{v}, \overset{r}{w})$ is a homogeneous function of order p in (x_1, x_2, x_3), of which the coefficients are rational integral functions of order q in v and order r in w, and if D is the discriminant of F with regard to (x_1, x_2, x_3), D the discriminant of Ω with regard to v, then D involving w is of the form

$$D = A \cdot B^2 \cdot C^3.$$

Taking $F(x_1, x_2, x_3) = 0$ as the equation to a curve, $\Omega = 0$ is the condition that the curve may have a double point.

$A = 0$ says for this double point $\frac{\partial F}{\partial x_1} = 0$. A is the discriminant of F with regard to (x_1, x_2, x_3) and v; $B = 0$ is the condition for two double points; $C = 0$ is the condition for a cusp.

For the values* I distinguish again, corresponding to cases 1° and 2° in Lemma I.:

1°. Let all coefficients of (x_1, x_2, x_3) be of order q in v, and the coefficients of v be of order r in w. Then

$$F(\overset{p}{x_1}, \overset{p}{x_2}, \overset{p}{x_3}; \overset{q}{v}; \overset{r}{w}) = (v, 1)^q (w, 1)^r (x_1, x_2, x_3)^p,$$

and Ω is of order $3q(p-1)^2$ in v, $3r(p-1)^2$ in w,

A	"	$6r(p-1)(2pq-p-q+1)$ in w,
B	"	$3q r(p-1)(p-2)(3q^2-3q-11)$ in w,
C	"	$24q r(p-1)(p-2)$ in w.

2°. All coefficients of (x_1, x_2, x_3) are of order q in v, but the coefficient of v^q is either of order k or if under $q-k$ in w. F is of form

$$F(\overset{p}{x_1}, \overset{p}{x_2}, \overset{p}{x_3}; \overset{q}{v}, \overset{r}{w}).$$

In the above orders of A, B, C, substitute $\frac{1}{2}q$ for r.

* For the manner in which these cases are determined, see Art. 3, footnote.

On Unicursal Series of Curves.

3. The method of determining the singularities of the envelope of a series of curves, which I am going to explain, is directly applicable to unicursal series only, and of these alone I shall treat. By a unicursal series of curves, I understand a series of such a nature that to each point on a straight line one curve corresponds, and, *vice versâ*, to each curve one point on the line. Instead of points on a line, we may of course take points on a unicursal curve, or lines in a pencil, &c. Such a series is algebraically defined by the fact, that all curves of the series are represented by an equation $(\lambda)^r (x_1, x_2, x_3)^m$ which is rational and integral with respect to one parameter (λ). Or this equation contains more parameters, which are connected by a unicursal parametric locus. The r^{th} polars of points on a unicursal curve, with respect to any fundamental curve, furnish an instance of such a series.

With regard to this point I refer to Professor Cayley's paper "On the Curves which satisfy given conditions," Annex. I., *Philosophical Transactions*, 1867.

In case that the unicursal parametric locus is a curve, this is supposed to be without a cusp. Otherwise the numbers given in the following, or at least some of them, require a correction.

4. Let us suppose a unicursal series of curves $(\lambda)^r (x_1, x_2, x_3)^m$ of order m and of index r, with parameter λ, and let us cut it by a pencil of lines. If θ be the parameter of this pencil, then a constant value of θ fixes a line in the pencil, as a transversal L of the curves, and a point on this transversal may be determined by a variable x. Between x and λ exists an equation of order r in λ and of order m in x, because to each value of λ corresponds one curve which cuts the transversal in m points x, and through each point x pass r curves, corresponding to r values of λ. The parameter θ of the pencil enters in the degree m in such a way that the equation is of the form

$$F\overset{r}{(}\lambda\,;\,\overset{m}{\overline{x, \theta}}) = 0,$$

as treated in Lemma I, 2°.

Take the discriminant Ω with regard to λ, and the discriminant $D = A \cdot B^2 \cdot C^3$ of Ω with regard to x; then Ω is a function of x and θ, and A, B, C are functions of θ.

Ω gives for constant θ points x on the transversal L, through which two consecutive curves of the series pass, or it determines points x on L, where the envelope of the series cuts L. The order of Ω in x gives therefore the order of the envelope. This is a well known method, and used for instance by Cremona, in his "Introd. ad una teoria geom. &c."

If we do not cut our series by a transversal, then Ω becomes a function of the coordinates, the equation to the envelope, in accordance with the rules given in the differential calculus.

$D = 0$ determines such values of θ, or such positions of the transversal L, that two points x, where the transversal cuts the envelope, coincide. L becomes either a common tangent to the envelope; here one value of x and one value of λ correspond to this θ, which is a root of $A = 0$; or the line L passes through a double point of the envelope; then to this value of θ and the corresponding x two values of λ belong, and θ is a root of $B = 0$; or, lastly, the transversal L passes through a cusp, and θ is a root of $C = 0$. Consequently the orders of A, B, C determine the class, the number of double points, and the number of cusps, respectively, of the envelope. Making use of Plücker's formulæ:—

The envelope of a unicursal series of curves, viz. $(\lambda)^r (x_1, x_2, x_3)^m = 0$, *of order m and index r, is*

of order $2m(r-1)$,

of class $m[3mr-2(m+r)+2]$.

It has

$2m^2(r-2)(r-3)$ *double points*,

$3m^2(r-2)$ *cusps*,

$12m(m-1)(r-1)$ *tangents of inflection*,

$\frac{1}{2}m\{m[3mr-2(m+r)+2]^2-m(3r-2)-36(m-1)(r-1)\}$ *double tangents.*

5. The discriminant Ω' of $F(\overset{r}{\lambda}; \overset{m}{x}, \theta)$ with regard to x, and the discriminant $D' = A' . B'^2 . C'^3$ of Ω' with regard to λ, contain some other interesting results. The orders are again determined by Lemma I., 2°. $\Omega' = 0$ determines for constant θ such values of the parameter λ, that the corresponding curve has two coincident points on the transversal L; hence it determines curves which touch a fixed line. A' is the discriminant of F with regard to x and λ, therefore identical with A. $B' = 0$ gives such values of θ, that two double roots x of F exist for one value of λ, or it determines lines in the pencil which are double tangents to a curve in the series. $C' = 0$ determines, in the same way, lines in the pencil which are tangents of inflection to a curve in the series. Hence

In a unicursal series of curves $(\lambda)^r (x_1, x_2, x_3)^m = 0$, *of order m and index r, there are*

$2r(m-1)$ *curves which touch a given line*,

$2mr(m-2)(m-3)$ *curves which have a double tangent passing through a given point*,

$3mr(m-2)$ *curves which have a tangent of inflection passing through a given point.*

This includes:

The double tangents of a unicursal series of order m and index r envelope a curve of class $2mr(m-2)(m-3)$,

and the tangents of inflection envelope a curve of class

$$3mr(m-2).$$

I now leave the general theory of series, and take series of polar curves to establish other theorems, some of which may be easily extended to series in general.

Singularities of Steinerians.

6. Steiner, in his treatise "Allgemeine Eigenschaften der algebraischen Curven" (*Crelle's Journal*, vol. 47), has considered some covariant curves, which he calls "Conjugirte Kerncurven." One of these curves is the locus of poles of first polars which have a double point, and the second or conjugate curve is the locus of these double points. The first of these curves is called by Cremona the "Steinerian" of the fundamental curve, and the second is the "Hessian."

Analogous curves exist for the other polars, and of these I have treated in my paper, read before the Society Nov. 12, 1868. There I used the symbol $S^{(r)}$ to denote the locus of poles the r^{th} polars of which have a double point, or the locus of double points of $(n-r-1)^{\text{th}}$ polars. Then $S^{(n-r-1)}$ is the locus of double points of r^{th} polars and also the locus of poles of $(n-r-1)^{\text{th}}$ polars which have a double point. The curves $S^{(r)}$ and $S^{(n-r-1)}$, where n is the order of the fundamental curve, are conjugate and correspond point for point. Henceforth I shall call all these curves "Steinerians," $S^{(r)}$ the "r^{th} *Steinerian of the fundamental curve*," $S^{(r)}$ and $S^{(n-r-1)}$ "*conjugate Steinerians.*"

Analytically, if U be the fundamental curve of order n, the equation to an r^{th} polar of pole (y_1, y_2, y_3) is the r^{th} emanant of U,

$$(y_1\delta_{x_1}+y_2\delta_{x_2}+y_3\delta_{x_3})^r \mathrm{U} = 0,$$

and this is of form $f(\overbrace{x_1, x_2, x_3}^{n-r}; \overbrace{y_1, y_2, y_3}^{r}) = 0.$

Then $S^{(r)}$ is the discriminant of f with regard to the original coordinates x_1, x_2, x_3. Or, *the* r^{th} *Steinerian* $S^{(r)}$ *is the discriminant with regard to the original variables of the* r^{th} *emanant of* U.

In the paper cited, I have determined the order and class of $S^{(r)}$ and developed some properties of these curves. The singularities are now easily found by Lemma II., and those properties too are contained in Lemmas I. and II., when interpreted in different ways.

Introducing again, for the coordinates y_1, y_2, y_3, two variables y, λ, where λ is the parameter of a pencil of lines L, and y a quantity which determines a point on the line L, the equation

$$f(\overbrace{x_1, x_2, x_3}^{n-r}; \overbrace{y_1, y_2, y_3}^{r}) = 0$$

to the r^{th} polar of (y) changes into one of the form

$$F(\overbrace{x_1, x_2, x_3}^{n-r}; \overbrace{y, \lambda}^{r}) = 0.$$

The discriminant Ω of this equation with regard to (x_1, x_2, x_3) determines poles y, the r^{th} polars of which have double points, or it determines for constant λ the points where the line L cuts the r^{th} Steinerian $S^{(r)}$. The discriminant $D = A.B^2.C^3$ of Ω with regard to y determines lines L in the pencil, which are either tangents to $S^{(r)}$, or pass through a double point or through a cusp of $S^{(r)}$. Writing

$$m = n - r - 1,$$

so that $S^{(r)}$ and $S^{(m)}$ are conjugate Steinerians, we have, by Lemma I., 2°,

The r^{th} Steinerian of a fundamental curve of order n has

the order	$3rm^2$,
the class	$3rm\,(2mr+r-m)$,
number of double points	$\frac{3}{2}r^2m\,(m-1)(3m^2+3m-11)$,
,, *cusps*	$12r^2m\,(m-1)$,
,, *tangents of inflection*	$3mr\,(10mr-r-6m)$,
,, *double tangents*	$\frac{3}{2}mr\,\{3mr\,(2mr+r-m)^2 - 32mr+2r+18m\}$.

Interchanging m and r, we get the singularities of $S^{(m)}$.

These numbers agree, for $r=1$, with those given by Steiner, Cremona, and Clebsch, for $S^{(1)}$, which is Cremona's Steinerian. The orders in Theorem II. are in fact determined by aid of this special case.*

On Polar Curves.

7. Speaking of polars without stating especially with regard to which curve they are taken, I mean always polars with regard to a fundamental curve of order n, having no double points and cusps. If a pole (y) moves on a unicursal curve of order m without cusps, then the r^{th}

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* In Lemma II. 1°, let α, β, γ be the orders of A, B, C in coefficients of x_1, x_2, x_3, v for $q=1$, or the orders in w for $q=1, r=1$. Then the orders of A, B, C, in w will be $r\alpha, r\beta, r\gamma$, in case the coefficients are functions of order r in w.

Let $D' = A'.B'^2.C'^3$ be the discriminant of Ω with regard to w, then, as in Lemma I., B, C are of the same order in w as B′, C′ in v. The order of B′ in v for $q=1$ is therefore $r\beta$, and this is at the same time the order of B′ in coefficients of x_1, x_2, x_3, w. Hence, if these coefficients be of order q in r, B′ will be of order $qr\beta$ in r, which again is the order of B in w. In the same way we find the order of C is $pr\gamma$ for the general case 1°. β and γ are independent of q and r.

In case 2°, where the orders in w of the coefficients form an arithmetical progression, the orders of B, C in w are $q^2\frac{\beta}{2}$, $q^2\frac{\gamma}{2}$. Now $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ are the numbers of double points and cusps of $S^{(1)}$, and therefore known, whence the orders of B, C in general. The order of A is afterwards ascertained by these orders and the order of D, which is easily determined. This gives for A, in case 2°, the same order, which I found in a different way to be the class of $S^{(r)}$.

polars form a unicursal series of order $n-r$ and index mr. The envelope of this series is the r^{th} polar of the unicursal curve. Hence, by the general theorem, Art. 4:—

The r^{th} polar of a unicursal curve of order m without cusps is
of order $2(n-r)(mr-1)$,
of class $(n-r)\{(n-r)(3mr-2)-2(mr-1)\}$.

It has $2(n-r)^2(mr-2)(mr-3)$ *double points*,
$3(n-r)^2(mr-2)$ *cusps*,
$12(n-r)(n-r-1)(mr-1)$ *tangents of inflection*,
$\frac{1}{2}(n-r)\{(n-r)[(n-r)(3mr-2)-2(mr-1)]^2-(n-r)(3mr-2)$
$-36(n-r-1)(mr-1)\}$ *double tangents.*

Cremona, in his "Introd. ad una teor. geom.," Art. 17, 103, determines the singularities of the $(n-1)^{\text{th}}$ polar of a curve of order m, which has, quite general, δ double points and κ cusps. This curve becomes unicursal for $\delta+\kappa=\frac{(m-1)(m-2)}{2}$. But it is only in case $\kappa=0$ that my number, for $r=n-1$, agrees with those. As soon as the unicursal curve has cusps, my numbers require a correction. I suppose, therefore, in this paper, unicursal curves to be without cusps.

Further, by Art. 5:

If a pole moves on a unicursal curve of order m without cusps, then the double tangents of its r^{th} polars envelope a curve of class

$$2mr(n-r)(n-r-2)(n-r-3),$$

and the tangents of inflection envelope a curve of class

$$3mr(n-r)(n-r-2).*$$

8. The equation to the r^{th} polar of a pole (y) is of form

$$f(\overbrace{x_1, x_2, x_3}^{n-r}; \overbrace{y_1, y_2, y_3}^{r}) = 0.$$

If, again, the pole (y) lie on a line L belonging to a pencil with parameter λ, and if y determine a point on this line; if, further, the polar be cut by a transversal R, on which x determines a point, then the equation changes into the form

$$F(\overbrace{x}^{n-r}; \overbrace{y, \lambda}^{r}) = 0.$$

We may treat y as the pole and x as a point on the r^{th} polar of y, or x as the pole and y as a point on the $(n-r)^{\text{th}}$ polar of y. This gives rise to a number of reciprocal theorems, some of which I shall state, although they are partly known. The discriminant Ω of F with regard

* This theorem holds also in case the curve of order m is not unicursal. Compare foot-note to Art. 9.

to x determines for a constant λ poles y on L, the r^{th} polars of which touch the line R; or it determines points y, where the $(n-r)^{\text{th}}$ polar of the line R cuts the line L. Or

On any line L *are* $2r(n-r-1)$ *poles* y, *the* r^{th} *polars of which touch any other line* R *at points* x. *These points* x *are the poles of* $(n-r)^{th}$ *polars which touch the* $(n-r)^{th}$ *polar of* R *at the points* y, *where* L *cuts this polar.*

The line L *cuts the* $(n-r)^{th}$ *polar of the line* R *in* $2r(n-r-1)$ *points* y, *and these points are the poles of the* r^{th} *polars, which touch the line* R *at points* x.

The discriminant $D = A \cdot B^2 \cdot C^3$ of Ω with regard to λ gives, when interpreted in a similar way,—

Through any given point pass $r(3rn-3r^2-2n+2)$ *lines* L, *the* r^{th} *polars of which touch a given line* R.

Through any given point pass $r(3rn-3r^2-2n+2)$ *lines* L, *which touch the* $(n-r)^{th}$ *polar of a given line* R.

Any given line R *is a double tangent for*

$$2r^2(n-r-2)(n-r-3)$$

r^{th} *polars of points* y, *and tangent of inflection for* $3r^2(n-r-2)$ r^{th} *polars of points* y'.

The $(n-r)^{th}$ *polar of a given line* R *has*

$$2r^2(n-r-2)(n-r-3)$$

double points y *and* $3r^2(n-r-2)$ *cusps* y'.

This reciprocity may also be stated thus:—

If the r^{th} *polar of a line* L *touches a second line* R, *then the* $(n-r)^{th}$ *polar of* R *touches* L.

If a line R *be a double tangent or tangent of inflection to the* r^{th} *polar of a point* y, *then this point* y *is a double point or cusp respectively of the* $(n-r)^{th}$ *polar of* R.

9. Let us now take R a line in a pencil with parameter λ, and abolish the line L altogether. We obtain an equation

$$F(\overbrace{x, \lambda}^{n-r}; \overbrace{y_1, y_2, y_3}^{r}) = 0.$$

Its discriminant Ω with regard to x represents for constant λ the $(n-r)^{\text{th}}$ polar of R, and for variable λ a series of these polars of order $2r(n-r-1)$ and of index $(n-r)(n-r-1)$. Or, interchanging r and $n-r$:—

The r^{th} *polars of lines passing through a fixed point form a series of order* $2(n-r)(r-1)$ *and of index* $r(r-1)$.

Each of these curves contains a certain number of double points and cusps, which form part of the envelope of the series. This envelope consists, therefore, of the locus of double points, of the locus of cusps,

and of the envelope proper. The discriminant $D = A \cdot B^2 \cdot C^3$ of Ω with regard to the parameter λ is the equation to this envelope. We may again treat (x) as pole or (y) as pole. For (x) as pole, $B = 0$ is the equation to the locus of double points of $(n-r)^{\text{th}}$ polars of lines in a pencil. For (y) as pole, B represents the locus of poles of r^{th} polars for which the lines R in the pencil are double tangents. Similarly for C. Hence

The double points of $(n-r)^{th}$ polars of lines in a pencil form a curve of order

$$2r\,(n-r)\,(n-r-2)\,(n-r-3),$$

which is at the same time the locus of poles of r^{th} polars, for which the lines in the pencil are double tangents.

The cusps of $(n-r)^{th}$ polars of lines in a pencil form a curve of order

$$3r\,(n-r)\,(n-r-2),$$

*and this is at the same time the locus of poles of r^{th} polars, for which the lines in the pencil are tangents of inflection.**

The orders are determined by Lemma I., 3°, to which our case belongs.

10. The factor A, which represents the envelope proper, breaks up into two factors, $A = f(\overbrace{\xi_1, \xi_2, \xi_3}^{n-r}; \overbrace{y_1, y_2, y_3}^{r}) \cdot \Delta$.

The first factor is the $(n-r)^{\text{th}}$ polar of the centre (ξ) of the pencil. Δ is the discriminant of f with regard to x_1, x_2, x_3; hence it is the locus of poles of r^{th} polars with a double point, or Δ is the r^{th} Steinerian $S^{(r)}$ of the fundamental curve. Interpreting $A = 0$ for (y) as pole, we find that it contains those poles of r^{th} polars which have a double point on a line R.

The envelope of the $(n-r)^{th}$ polars of lines in a pencil consists of the $(n-r)^{th}$ polar of the centre of the pencil and of the r^{th} Steinerian $S^{(r)}$.

This gives a new definition for the Steinerians:

The Steinerian $S^{(r)}$ is the envelope of the $(n-r)^{th}$ polars of lines in a pencil, whatever the position of this pencil may be. Hence it is also the

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* On a curve of order m are therefore

$$2mr\,(n-r)\,(n-r-2)\,(n-r-3)$$

poles the r^{th} polars of which have a double tangent passing through a fixed point, and there are

$$3mr\,(n-r)\,(n-r-2)$$

poles the r^{th} polars of which have a tangent of inflection passing through a given point. In other words, if a pole moves on any curve of order m, then the double tangents and tangents of inflection of its r^{th} polars envelope curves of class

$$2mr\,(n-r)\,(n-r-2)\,(n-r-3) \quad \text{and} \quad 3mr\,(n-r)\,(n-r-2).$$

These are the same numbers which were given in Art. 7 for the case that the pole moves on a unicursal curve of order m.

locus, where the $(n-r)^{th}$ *polars of all lines touch each other. The* $(n-r)^{th}$ *polars of lines (and of curves in general), passing through a point on the conjugate Steinerian* $S^{(n-r-1)}$, *touch at the same point on* $S^{(r)}$, *the one corresponding to that point on* $S^{(n-r-1)}$. But it must be remembered that the $(n-r)^{th}$ polars of lines in a pencil envelope, besides, the r^{th} polar of the centre of pencil.

The last part of the proposition follows immediately from the relation between conjugate curves. This relation gives rise further to the following theorems, where only $n-r$ and r are interchanged:

The r^{th} *polar of any line touches* $S^{(n-r)}$ *in* $3(r-1)(n-r)^2$ *point's, which correspond to the points where the line cuts the conjugate curve* $S^{(r-1)}$, *and it cuts it besides in* $6(n-r)^2 . r(r-1)(r-2)$ *points.*

Reciprocally,

On $S^{(r)}$ *are* $3r^2(n-r-1)$ *poles the* r^{th} *polars of which have a double point on a given line, and besides*

$$6r^2(n-r)(n-r-1)(n-r-2)$$

poles, the r^{th} *polars of which have a proper contact with the line.*

Analogous theorems hold for polars of curves instead of lines. We have,—

If we take the r^{th} *polars of curves in a series, then the Steinerian* $S^{(n-r)}$ *is part of the envelope of this new series.*

The r^{th} *polar of any curve* K *touches* $S^{(n-r)}$ *in points which correspond to the points of intersection of* K *and the conjugate curve* $S^{(r-1)}$.

If the curve K be of order n and have δ double points and κ cusps, then its r^{th} polar touches $S^{(n-r)}$ in $3m(r-1)(n-r)^2$ points, and cuts it besides in

$$3(r-1)^2(n-r)^2\{m(m+2r)-(2\delta+3\kappa)\}-3m^2(r-1)(n-r)^2$$

points, the poles of $(n-r)^{th}$ polars, which touch K.

Further,

The r^{th} *polar of a curve* K *cuts* $S^{(n-r-1)}$ *in points which correspond to those where* K *cuts the conjugate curve* $S^{(r)}$;

because a point where K cuts $S^{(r)}$ is double point of an r^{th} polar, and therefore on the r^{th} polar of K.

Mixed Polars of Points and Curves.

11. Let (y) be a point on a curve $C^{(m)}$ of order m, and (z) a point on a curve $C^{(\mu)}$ of order μ. If (y) moves along $C^{(m)}$, then the r^{th} polar of (y) envelopes a curve, which is known as the r^{th} polar of $C^{(m)}$. In the same way, the r^{th} polar of (y) with regard to the ρ^{th} polar of (z), or "*the mixed* $(r, \rho)^{th}$ *polar of the two poles* (y, z)," envelopes a curve if (z)

moves on $C^{(m)}$, which may be called "*the mixed* $(r, \rho)^{th}$ *polar of the curve* $C^{(m)}$ *and the point* (z)"; it is the r^{th} polar of $C^{(m)}$ with regard to the ρ^{th} polar of (z). This definition determines the singularities, if $C^{(m)}$ be unicursal.

Now let (z) too move along the curve $C^{(\mu)}$, then the last mentioned polar will envelope a new curve, which is to be called, in conformity with the above terminology, "*the mixed* $(r, \rho)^{th}$ *polar of the two curves* $C^{(m)}$, $C^{(\mu)}$." For this polar we have therefore the definition,—

The mixed $(r, \rho)^{th}$ *polar of two curves* $C^{(m)}$, $C^{(\mu)}$ *is the envelope of the* r^{th} *polar of* $C^{(m)}$ *with respect to the* ρ^{th} *polar of a variable point on* $C^{(\mu)}$, or, as is easily seen, *the envelope of the* ρ^{th} *polar of* $C^{(\mu)}$ *with respect to the* r^{th} *polar of a variable point on* $C^{(m)}$.

The mixed $(r, \rho)^{th}$ *polar of* $C^{(m)}$, $C^{(\mu)}$ *is also the mixed* $(\rho, r)^{th}$ *polar of* $C^{(\mu)}$, $C^{(m)}$.

The order of such a mixed polar of two curves can be determined by Lemma I., in case the two curves be unicursal and without cusps.

12. The equation to the mixed $(r, \rho)^{th}$ polar of two poles (y, z) is of the form

$$f(\overbrace{x_1, x_2, x_3}^{n-r-\rho};\ \overbrace{y_1, y_2, y_3}^{r};\ \overbrace{z_1, z_2, z_3}^{\rho}) = 0.$$

If (y) be on a unicursal curve $C^{(m)}$, and (z) on a unicursal curve $C^{(\mu)}$, we may express y_1, y_2, y_3 as rational integral functions of order m of one parameter y, and z_1, z_2, z_3 as such functions of order μ of another parameter z. If we cut, besides, the polar by a transversal, we may express x_1, x_2, x_3 by a variable x. Then we obtain, for the points x where the transversal cuts the polar, an equation

$$F(\overset{n-r-\rho}{x},\ \overset{mr}{y},\ \overset{\mu\rho}{z}) = 0.$$

This equation determines for constant z points x where the transversal cuts the series of mixed polars of (y, z), and its discriminant with regard to y determines the points where the transversal cuts the envelope of this series or the mixed polar of $C^{(m)}$ and (z). The orders of discriminants are to be taken from Lemma I., 1°. Hence

The mixed $(r, \rho)^{th}$ *polars of* $C^{(m)}$ *and a variable point* (z) *on* $C^{(\mu)}$ *form a series of order* $2(n-r-\rho)(mr-1)$ *and of index* $2\rho\mu(mr-1)$.

The discriminant $D = A.B^2.C^3$ of Ω with regard to x gives

In the series of mixed $(r, \rho)^{th}$ *polars of* $C^{(m)}$ *and points* (z) *on* $C^{(\mu)}$ *are*

$$2\mu\rho\{3mr(n-r-\rho) - 2(mr+n-r-\rho-1)\}$$

curves which touch a given line.

The double points in the series form a curve of order

$$4\mu\rho(n-r-\rho)(mr-2)(mr-3),$$

and the cusps in the series form a curve of order

$$6\mu\rho(n-r-\rho)(mr-2).$$

13. The discriminant $D' = A' . B'^2 . C'^3$ of Ω with regard to z determines the envelope of our series. It breaks up into the loci of double points and cusps, which have just been determined, and into the envelope proper, which is given by $A' = 0$.

The mixed $(r, \rho)^{th}$ polar of two unicursal curves C^m, C^μ without cusps, of orders m and μ, is of order

$$2(n-r-\rho)\{3m\mu r\rho - 2(mr+\rho\mu)+2\}.$$

This gives, for $m = \mu = 1$, $r = \rho = 1$, the order $2(n-2)$ of the mixed second polar of two lines, which is right according to Cremona.* This mixed second polar of two lines appears here as the envelope of a series of curves of order zero and index zero. Cremona defines it, indeed, as a locus, but his definition cannot be extended, whilst the definition here given is easily extended to mixed polars of three or more curves.

A' is the discriminant of F with regard to y and z. If we do not cut the series by a transversal, we have the equation

$$F(\overset{n-r-\rho}{x_1, x_2, x_3};\ \overset{mr}{y};\ \overset{\rho}{z}) = 0,$$

and then A' is the equation to that mixed polar.

To form the equation to the mixed $(r, \rho)^{th}$ polar of two unicursal curves, take the equation to the mixed $(r, \rho)^{th}$ polar of two points, one on each curve, and express the coordinates of these points as rational integral functions of two parameters; then the discriminant of this equation with regard to the two parameters gives the required equation.

14. To determine the class and singularities of this mixed polar, we have to introduce a parameter λ of our transversal so that this belongs to a pencil. Thus we obtain an equation

$$F(\overset{n-r-\rho}{x, \lambda};\ \overset{mr}{y};\ \overset{\rho}{z}) = 0.$$

Its discriminant A' with regard to y, z gives points of the mixed polar on the transversal λ. The discriminant D'' of A with regard to x is function of λ, and breaks up again into portions $D'' = A'' . B''^2 . C''^3$.

A'' determines transversals λ which touch the mixed polar, and B'', C'' transversals which pass through a double point or a cusp. Their orders are not given by the two Lemmas, and I have determined only the order of A'',† or the class of our mixed polar.

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* Introd. ad una teor. geom. &c., Art. 21, 125.

† A'' is the discriminant of F with regard to x, y, z. I find the discriminant of a quantic $F(\overset{p}{x}, \overset{q}{y}, \overset{r}{z}) = (x, U^p)^q (x, U^q)^r$, with regard to x, y, z, of order

$$2(12s_3 - 6s_2 + 4s_1 - 4),$$

where s_1 is the sum of p, q, r; s_2 the sum of their products taken two together, and s_3 their product.

If the coefficients of x, y, z be functions of λ of order α, we have, of course, to

The mixed $(r, \rho)^{th}$ polar of two unicursal curves C^m C'^μ of orders m and μ is of class

$$(n-r-\rho)\{2m\mu r\rho(n-r-\rho)-6[mr\mu\rho+(mr+\mu\rho)(n-r-\rho)] + 4(mr+\mu\rho+n-r-\rho)-4\}.$$

For the second mixed polar of two lines we get thus the class $2(n-2)(2n-5)$, which is right.

15. Let
$$F(\overset{x}{x}, \overset{\eta}{\eta}; \overset{y}{y}, \overset{\lambda}{\lambda}; \overset{\rho}{z_1}, \overset{\rho}{z_2}, \overset{\rho}{z_3}) = 0$$
be the equation to the mixed $(r, \rho)^{th}$ polar of the points (y, z), where x determines a point on a line R in a pencil with parameter η, and y a point on a line L in a pencil with parameter λ. Take discriminants Ω with regard to x, and $D = A \cdot B^2 \cdot C^3$ of Ω with regard to y.

$\Omega = 0$ gives poles y on a line L (for a constant λ), so that the mixed $(r, \rho)^{th}$ polar of (y, z) touches a line R (for a constant η).

$A = 0$, as function of λ, determines lines L in a pencil so that the mixed $(r, \rho)^{th}$ polar of the line L and point (z) touches a line R. Or it determines for variable η lines R, which touch the mixed $(r, \rho)^{th}$ polar of (L, z). Hence,

Through any point $r\{3r(n-r-\rho)-2(n-\rho)+2\}$ lines L may be drawn, that the mixed $(r, \rho)^{th}$ polar of (L, z) touches a given line R.

The poles (z), for which the mixed $(r, \rho)^{th}$ polar of (L, z) touches a given line R, are on a locus of order

$$2\rho\{3r(n-r-\rho)-2(n-\rho)+2\}.$$

This locus is also the envelope of the r^{th} polar of L with regard to the $(n-r-\rho)^{th}$ polar of a variable point on R, or it is the mixed $(n-r-\rho, r)^{th}$ polar of the lines (R, L).

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n^{th} pole by ρ to obtain the order of the discriminant in λ. But if the orders of these coefficients form an arithmetical progression $0, 1, \ldots\ldots \rho$, we have to multiply by $\frac{1}{2}\rho$ only to obtain the order of the discriminant. Hence the order of A'' in the text.

The above expression for the order of the discriminant of $F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w})$ may be written in the form
$$|4\,s_3 - 2\,|3\,s_2 + 2^2\,|2\,s_1 - 2^3.$$

The discriminant of $F(\overset{p}{u}, \overset{q}{v})$ with regard to u, v is, according to Lemma I., 1°, (order of A) of order
$$6pq - 4(p+q) + 4 = |3\,s_2 - 2\,|2\,s_1 + 2^2;$$
and for the discriminant of $F(\overset{p}{u})$ we have the order
$$2(p-1) = |2\,s_1 - 2.$$

From this it seems to follow that the order of discriminant of a quantic
$$F(\overset{p}{u}, \overset{q}{v}, \overset{r}{w}, \overset{m}{x}, \ldots) = (u, 1)^p (v, 1)^q (w, 1)^r (x, 1)^m \ldots\ldots$$
containing ν variables, will be
$$|\nu+1\,s_\nu - 2\,|\nu\,s_{\nu-1} + 2^2\,|\nu-1\,s_{\nu-2} - \ldots\ldots \pm 2^\nu s_0,$$
where s_k is the sum of products of $p, q, r, \ldots$ taken k together, and where $s_0 = 1$.

The last proposition is the interpretation of A, if we take x and y as poles moving on two fixed lines R and L.

16. $B=0$, $C=0$ say,

A given line R is double tangent for

$$2r^2(n-r-\rho-2)(n-r-\rho-3)$$

mixed $(r,\rho)^{th}$ *polars of points* (y, z), z *being a fixed point, and*

The line R *is tangent of inflection for* $3r^2(n-r-\rho-2)$ *such mixed polars.*

If the pole y *lie on a fixed line* L, *then the locus of a pole* (z) *such that the mixed* $(r,\rho)^{th}$ *polar of* (y, z) *has a given line* R *as double tangent, is of order* $4\rho r(n-r-\rho-2)(n-r-\rho-3)$.

The locus of (z), *if* R *be a tangent of inflection, is of order*

$$6\rho r(n-r-\rho-2).$$

These last loci determine at the same time the poles (z) *for which the mixed* $(n-r-\rho, \rho)^{th}$ *polar of the line* R *and the point* (z) *have a double point or a cusp respectively, on a fixed line* L.

And they are the loci of the double points and cusps respectively of the mixed $(n-r-\rho, r)^{th}$ *polars of* (x, y), (x) *being a point on* R, (y) *on* L.

Steinerians of Polars.

17. The ρ^{th} Steinerian of the r^{th} polar of (y) is the locus of poles (z), so that the ρ^{th} polar of (z) with regard to the r^{th} polar of (y) or the $(\rho, r)^{th}$ polar of (z, y) may have a double point. This is identical with saying that the mixed $(r,\rho)^{th}$ polar of (y, z) may have a double point. Hence the equations to the ρ^{th} Steinerian of the r^{th} polar of (y), and to the r^{th} Steinerian of the ρ^{th} polar of (z), will be identical in form; only in the first case (z) is treated as the variable point, and in the second (z) is constant and (y) is variable. I denote the ρ^{th} Steinerian of the r^{th} polar of (y) by $S_{y^r}^{(\rho)}$ or by $S_{y^r}^{(\rho)}(z)$. Then $S_{y^r}^{(\rho)}(z)$ and $S_{z^\rho}^{(r)}(y)$ are identical in form.

To each pole belongs a number of Steinerians, which have reciprocal relations similar to those of polars.

If the pole, with regard to which the Steinerians are taken, moves on a line, then the Steinerians constitute a unicursal series, of which I shall treat. The order and singularities of these Steinerians follow from those of Steinerians in general (Art. 6).

18. The equation to the mixed $(r,\rho)^{th}$ polar of (y, z) may be written in the form

$$F(\overbrace{x_1, x_2, x_3}^{n-r-\rho};\ \overbrace{y, \lambda}^{r};\ \overbrace{z_1, z_2, z_3}^{\rho}) = 0,$$

where the pole (y) is supposed to be on a line in a pencil with parameter λ. Or it may be written in the form

$$F(\overbrace{x_1, x_2, x_3}^{n-r-\rho};\ \overbrace{y_1, y_2, y_3}^{r}:\ \overbrace{z, \mu}^{\rho}) = 0,$$

where the pole (z) is supposed to lie on a line in a pencil with parameter μ.

The discriminant Ω of this with regard to x_1, x_2, x_3 gives the condition, that this mixed polar may have a double point. Hence Ω is the quantic $S_{y^r}^{(\rho)}(z)$ or $S_{z^\rho}^{(r)}(y)$ according as we choose the variables. I form $D = A \,.\, B^2 \,.\, C^3$ with regard to y for the first form of the equation to polar, and $D' = A' \,.\, B'^2 \,.\, C'^3$ with regard to z for the second equation. D is a function of λ and (z), D′ of μ and (y). The orders with regard to λ or μ are determined by Lemma II., 2°, and with regard to (z) or (y) by Lemma II., 1°.

Interpreting these discriminants in different ways, we obtain the following results, where always the factor of D or D′, which is considered, is added.

The curves $S_{y^r}^{(\rho)}(z)$, *or the* ρ^{th} *Steinerians of the* r^{th} *polars of points* (y) *on a straight line, form a unicursal series of curves*

of order $3\rho(n-r-\rho-1)^2$, [order of Ω in (z),]
of class $3\rho(n-r-\rho-1)\{2\rho(n-r-\rho)-n+r+1\}$, [A′ in μ],
*of parametric order** $3r(n-r-\rho-1)^2$, [Ω in (y),]
of parametric class $6r(n-r-\rho-1)\{2\rho(n-r-\rho)-n+r+1\}$, [A′ in (y)].

The envelope is of order†

$$6\rho(n-r-\rho-1)\{2r(n-r-\rho)-n+\rho+1\}, \quad [A \text{ in } (z)].$$

The locus of double points of order

$$3r\rho(n-r-\rho-1)(n-r-\rho-2)\{3(n-r-\rho)^2-3(n-r-\rho)-11\},$$
[B in (z), or B′ in (y)].

The locus of cusps of order

$$24r\rho(n-r-\rho-1)(n-r-\rho-2), \quad [C \text{ in } (z), \text{ or } C' \text{ in } (y)].$$

Further,

There exist for a fixed pole (y)

$$\tfrac{3}{2}\rho^2(n-r-\rho-1)(n-r-\rho-2)\{3(n-r-\rho)^2-3(n-r-\rho)-11\}$$
[B′ in (μ)]

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* *Parametric order* determines the index of the series, or the number of curves in the series, passing through a point; *parametric class* the number of curves touching a given line.

† For $n = 4$, $\rho = r = 1$, we have the order of this envelope $= 12$, in agreement with Prof. Clebsch, Ueber Curven 4^{ter} Ordnung, Crelle's Journ., vol. 59, § 3. p. 132.

poles (z), *so that the mixed* $(r, \rho)^{th}$ *polar of* (y, z) *may have two double points, and* $12\rho^2(n-r-\rho-1)(n-r-\rho-2)$ [C′ in (μ)] *poles* (z), *so that the same mixed polar may have a cusp.* These points are the double points and cusps of $S^{(\rho)}_{yr}$. Hence, if we move the pole (y) along a line, then these poles (z) describe the loci of double points and cusps of our series of curves $S^{(\rho)}_{yr}$, the orders of which have been given.

19. The Steinerians of polars are of importance for the study of the Steinerians of the fundamental curve, and for the theory of polar curves in general. I must reserve a fuller treatment of them to another occasion, but I shall mention a few theorems, which follow easily from what has been said of these curves.

If (y) *be a point on the* r^{th} *Steinerian* $S^{(r)}$ *of the fundamental curve, then the corresponding point on the conjugate Steinerian* $S^{(n-r-1)}$ *is a double point of every Steinerian* $S^{(\rho)}_{yr}$ *of the* r^{th} *polar of* (y).

$S^{(\rho)}_{yr}$ has in general a certain number of double points and cusps, but this number may be increased for certain positions of the pole (y), and this is the case if (y) be on $S^{(r)}$. It follows that $S^{(r)}$ must be a factor of what Prof. Cayley calls the "special discriminant" of $S^{(\rho)}_{yr}$. It contains other factors besides this.

The fact that the quantics $S^{(\rho)}_{yr}(z)$ and $S^{(r)}_{z^\rho}(y)$ are identical, contains the reciprocity,

If the ρ^{th} *Steinerian of the* r^{th} *polar of* (y) *passes through* (z), *then the* r^{th} *Steinerian of the* ρ^{th} *polar of* (z) *passes through* (y).

Taking the conjugate curves into consideration, we have similarly,

If (x) *be a double point on the mixed* $(r, \rho)^{th}$ *polar of* (y, z), *then* (y) *is a double point of the* $(\rho, n-r-\rho-1)^{th}$ *polar of* (z, x), *and* (z) *is a double point of the* $(n-r-\rho-1, r)^{th}$ *polar of* (x, y). Or, with regard to the Steinerians,

If (z) *be a point on the* ρ^{th} *Steinerian of the* r^{th} *polar of* (y), *and* (x) *the corresponding point on the conjugate Steinerian, then* (y) *is a point on the* r^{th} *Steinerian of the* $(n-r-\rho-1)^{th}$ *polar of* (x), *and* (z) *the corresponding point on the conjugate Steinerian;* (x) *is a point on the* $(n-r-\rho-1)^{th}$ *Steinerian of the* ρ^{th} *polar of* (z) *and* (y) *is the corresponding point on the conjugate Steinerian.*

For the same meaning of (x), (y), (z), we have, in symbols:

(z)	lies on	$S^{(\rho)}_{z^r}$	and on	$S^{(\rho)}_{x^{n-r-\rho-1}}$,
(y)	,,	$S^{(r)}_{x^{n-r-\rho-1}}$	,,	$S^{(r)}_{z^\rho}$,
(x)	,,	$S^{(n-r-\rho-1)}_{z^\rho}$	,,	$S^{(n-r-\rho-1)}_{y^r}$.

Further,

If the ρ^{th} *Steinerian of the* r^{th} *polar of a pole* (y) *pass through the pole* (y), *then* (y) *must be a point on the* $(r+\rho)^{th}$ *Steinerian of the fundamental curve, or if* $S_{yr}^{(\rho)}$ *pass through* (y), *then* (y) *must lie on* $S^{(r+\rho)}$.

If this be satisfied and (x) *be the point on the conjugate curve* $S^{(n-r-\rho-1)}$ *corresponding to* (y) *on* $S^{(r+\rho)}$, *then the two curves* $S_{yr}^{(n-r-\rho-1)}$, $S_{y\rho}^{(n-r-\rho-1)}$ *intersect at* (x), *and* $S_{x^{n-r-\rho-1}}^{(r)}$, $S_{x^{n-r-\rho-1}}^{(\rho)}$ *intersect at* (y).

The Chairman presented some of the Author's statements in a different form. Dr. Hirst commenced a sketch of results arrived at by the method of "Quadric Transformation," in a communication made by him at the British Association Meeting in 1865. These remarks were prompted by Mr. Clifford's diagram, exhibited at the April meeting of the Society. The President closed the business of the evening by drawing attention to a model, constructed by Dr. Henrici, of the cubic surface $xyz=1$, pointing out the lines of curvature and other singularities.

The following presents were made to the Society:—

"Crelle's Journal," 70er Band, 1es Heft.

"Ueber Systeme von Functionen mehrer Variabeln," from "Monatsbericht" for March, 1869; from the Author, Prof. L. Kronecker.

"Monatsbericht," Februar, 1869.

June 10th, 1869.

Prof. CAYLEY, President, in the Chair.

Sir William Thomson, F.R.S., was admitted into the Society by the President.

Prof. Hirst proceeded with the second part of his communication on "Quadric Transformation;" in the course of the Chairman's remarks on it, he defined "Skew antipoints." Imagine a tetrahedron BB′CC′ in which the opposite sides BB′, CC′ are at right angles to each other and to the line joining their middle points M, N; and in which, moreover, $(CN)^2+(NM)^2+(MB)^2=0$, (or, what is the same thing, the sides CB, CB′, C′B, C′B′ are each $=0$; the tetrahedron is, of course, imaginary; viz., the lines CC′, BB′ and points M, N, may be real, but the distances MB = MB′ and NC = NC′, may be one real and the other imaginary, or both imaginary, but they cannot both be real). The points B, B′ and C, C′ are said to be "Skew antipoints."

Prof. Sylvester then gave an explanation of some of the processes employed in his paper on "Reducible Cyclodes."

The following present was made to the Society :—

"Crelle, 70 Band, Zweites Heft."

The following paper, an account of which was given by the author on April 8th, could not be inserted in the account of the Proceedings of that day :—

On the Focal Properties of Homographic Figures. By Henry J. Stephen Smith, F.R.S., Savilian Professor of Geometry in the University of Oxford.

A.—Focal Properties of Two Homographic Plane Figures.

(1.) *Two Plane Figures in Perspective.*

We consider two plane figures Ω and ω in perspective with one another; we denote the centre of the perspective by S, and the axis of the perspective (or the line of intersection of the two planes) by $\Omega_1\omega_1$; we exclude the cases in which the straight lines at an infinite distance in the two planes are corresponding lines; *i. e.*, we suppose that the centre of perspective is not at an infinite distance, and that the planes are not parallel. Let OY, $o'y$ be the vanishing lines of the planes Ω and ω, or the straight lines which in the planes Ω and ω correspond to the straight lines at an infinite distance in the planes ω and Ω; the plane Ω is divided by OY into two regions (Ω_1) and (Ω_2); similarly, $o'y$ divides ω into two corresponding regions (ω_1) and (ω_2). Let (Ω_1) be that region of Ω in which $\Omega_1\omega_1$ is situated; then $\Omega_1\omega_1$ is also situated in (ω_1); and it will be seen that if P, p are corresponding points in the regions (Ω_1), (ω_1), the radii vectores SP, Sp are of the same sign; but if P, p are corresponding points in (Ω_2), (ω_2), the radii vectores SP, Sp are of opposite signs; or, in the language of some writers on perspective, (Ω_1) and (ω_1) are projections of one another, but (Ω_2) and (ω_2) are transprojections of one another.

(2.) *The Correspondence of Directions.*

If the positive and negative directions on any straight line in either of the planes Ω and ω are regarded as determined, the corresponding directions on the corresponding line are also determined; viz., if a point move in the positive direction on a straight line in either plane, its image in the other plane moves in the positive direction on the corresponding straight line. Hence, if P,Q are two points in the same region of Ω, and p, q their images, which are of course in the corresponding region of ω, the direction from P to Q along the finite segment PQ is of the same sign as the direction from p to q along the finite segment pq; but if P, Q are in opposite regions of Ω, so that the

finite segment PQ is divided internally by OY in the point A, the direction $p \infty q$ will correspond to the direction PAQ, and the directions of the finite segments PQ, pq will be of opposite signs. We may add that if A is any point whatever on OY, to the directions PA, QA there will correspond similar or dissimilar directions on the parallel straight lines which are the images of the lines PA, QA, according as P and Q are in the same region, or in different regions of Ω. And in particular, if in the plane Ω there be drawn any parallel to the vanishing lines, the corresponding line in the plane ω will also be parallel to the vanishing lines, but the corresponding directions on the two parallels will be similar or dissimilar, according as they lie in the regions (Ω_1), (ω_1), or in the regions (Ω_2), (ω_2).

Again, if in the plane Ω we consider one of the two directions of rotation round any point as positive, (say, for example, that direction of rotation which viewed from S appears right-handed,) the signs of the directions of rotation will thereby be fixed for each point of the plane ω; but for all points in the region (ω_1) that direction of rotation, which viewed from S is right-handed, will be positive, whereas for all points of the region (ω_2) the same direction of rotation must be considered negative; it being inconsistent with the perspective relation to regard one and the same direction of rotation as being positive for all points of the one plane, and also to regard one and the same direction of rotation as being positive for all points of the other plane.

(3.) *The Equiangular Points, or Foci.*

Through S draw two lines perpendicular to the planes which bisect the dihedral angle formed by the intersecting planes Ω and ω. Let these perpendiculars meet the plane Ω in F_1, F_2, and the plane ω in f_1, f_2; let also SF_2f_2 be perpendicular to that bisecting plane which lies in the same angle with S; then SF_1, Sf_1 are of the same sign, and F_1, f_1 lie in the regions (Ω_1), (ω_1) respectively; but SF_2, Sf_2 are of opposite signs, and F_2, f_2 lie respectively in the regions (Ω_2), (ω_2).

Since each of the lines F_1f_1, F_2f_2, is at right angles to the line of intersection of Ω and ω, and is besides equally inclined to those two planes, any dihedral angle of which the axis is either F_1f_1, or F_2f_2, is intersected in two equal rectilineal angles by the planes Ω and ω. We thus obtain the theorem:

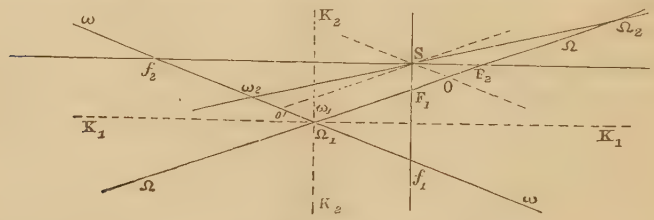
"Angles in the plane Ω at the points F_1, F_2, are projected into equal angles in the plane ω at the points f_1, f_2."

Or, more precisely,

"The angle contained by given directions on two straight lines lying in the plane Ω, and intersecting at F_1 or F_2, is equal to the angle formed in the plane ω by the corresponding directions on the corresponding straight lines, which intersect at f_1 or f_2."

For brevity, we shall call F_1, F_2, f_1, f_2, the *foci* of the perspective in the planes Ω and ω respectively. And since the directions of rotation, as viewed from S, are the same round F_1 and f_1, but are opposite round F_2 and f_2, we shall call F_1, f_1 the similar foci, and F_2, f_2 the dissimilar foci.

In the annexed diagram the plane of the paper is the plane of sym-



metry; *i.e.*, it is the plane passing through S, and cutting Ω, ω at right angles; $O\Omega_1$, $o'\omega_1$ are the traces of the given planes Ω and ω, K_1, K_2 are the traces of the bisecting planes; F_1F_2, f_1f_2 are the foci, which lie of course in the plane of symmetry; the axis of the perspective $\Omega_1\omega_1$ is perpendicular to the plane of the paper at Ω_1 or ω_1, and the vanishing lines OY, $o'y$ are perpendicular to the same plane at the points O, o', which we shall term the *centres* of the two planes. If Y, y are the points at infinity on the vanishing lines, Y and y are corresponding points; but it will be observed that the vanishing lines are not corresponding lines, nor the centres corresponding points. We may term the lines F_1OF_2, $f_1o'f_2$, which are corresponding lines at right angles to the vanishing lines, the focal axes of the two planes; so that the centre of each plane corresponds to the point at an infinite distance on the focal axis of the other. Since SO, So' are parallel to the traces of the planes Ω, ω, and SF_1f_1, SF_2f_2 to the traces of the bisecting planes, the figure $SO\Omega_1o'$ is a parallelogram, the triangles SOF_1, SOF_2, S$o'f_1$, S$o'f_2$, are isosceles, F_1F_2 and f_1f_2 are bisected at the centres O and o', SO is equal to OF_1 or OF_2, So' to $o'f_1$ or $o'f_2$; whence we find

$$\Omega_1F_1 = \omega_1f_1 = \tfrac{1}{2}(f_2f_1+F_2F_1),$$
$$\Omega_1F_2 = -\omega_1f_2 = \tfrac{1}{2}(f_2f_1-F_2F_1).$$

These equations, as all others in this paper, are to be interpreted, with regard to sign as well as magnitude; we shall avoid the use of the sign =, when we have to speak of equality irrespective of sign.

(4.) *The Equi-Segmental Axes, or Cyclic Lines.*

Any point of the line $\Omega_1\omega_1$, in which the planes Ω and ω intersect, considered as a point in either plane, has itself for its corresponding

point in the other plane. If we do not attend to the coincidence of the corresponding points, we may express this by saying that $\Omega_1\omega_1$, is an *equi-segmental* line in either plane; *i.e.*, that to any segment of $\Omega_1\omega_1$, considered as a line in either plane, an equal segment corresponds in the other plane. But besides this coincident pair of equi-segmental lines, there is another pair of corresponding equi-segmental lines which are not coincident. Through S extend a plane parallel to the plane containing the vanishing lines, and let it meet the focal axes of the two planes in Ω_2 and ω_2. The lines O_2Y, $\omega_2 y$, parallel to the vanishing lines, are equi-segmental lines. For if P, p are corresponding points on those lines, the radii vectores SP, Sp, SΩ_2, Sω_2, are respectively equal and of opposite signs, and the angles Ω_1SP, ω_1Sp, are equal; so that P, p lie at equal distances from the plane of symmetry, but on opposite sides of that plane; *i.e.*, the axes Ω_2Y, $\omega_2 y$ are equi-segmental. We may term the coincident axes $\Omega_1 y_1$, $\omega_1 y$, the similar axes, and the axes Ω_2Y, $\omega_2 y$ the dissimilar axes. Since $\Omega_1\Omega_2$ is double of Ω_1O, *i. e.* of $o'S$, or $o'f_1$, and similarly $\omega_1\omega_2$ is double of $\omega_1 o'$, *i.e.* of OS or OF_1, we see that in either plane the equi-segmental axes are situated symmetrically with respect to the centre of that plane, and that the semi-distance between the foci is equal to the semi-distance between the equi-segmental axes in the other plane. The semi-distance between the foci in either plane may conveniently be called the *parameter* of that plane. Designating the parameters of the planes Ω and ω by C and c, we shall find, if $C=c$, that the foci of each plane lie on its equi-segmental lines, and that the centre of perspective lies on one of the two bisecting planes. But if C and c are unequal, if for example $C<c$, the foci lie between the equi-segmental lines in Ω, and outside them in ω, and the distances $c+C$ and $c-C$, between a focus and the nearer and further equi-segmental line, are the same for both planes.

If we cause one of the two planes, for example the plane ω, to revolve round the axis of intersection of the two planes, the two figures, as is well known, will continue in perspective; and the locus of the centre of perspective will be a circle, lying in the plane of symmetry, and described on F_1F_2 as diameter. At one of the coincidences of the two planes which take place during a complete revolution, the similar foci come to coincide with one another and with the centre of perspective; and, in like manner, at the other coincidence, the dissimilar foci coincide with one another and with the centre of perspective; the similar foci continuing similar, and the dissimilar foci continuing dissimilar during the whole revolution. If, however, we bring together corresponding points in the dissimilar axes (which we may conceive done by causing either plane to rotate through an angle of 180° round an axis perpendicular to the plane $\Omega_2\omega_2$ at S), the two figures will again be in perspective, but the foci, which were before similar, will become dis-

similar, and *vice versâ*. Thus the two foci, and the two equi-segmental axes, in either plane, are not absolutely distinguished as similar to, or dissimilar from, their corresponding foci or axes; these denominations being, in fact, relative to one or other of the two ways in which the planes can be placed in perspective with one another, and changing when we pass from one of those ways to the other. In every case, if a focus or axis be regarded as similar, the nearer axis or focus is also to be regarded as similar.

The equi-segmental lines may, perhaps, be called the cyclic lines. This denomination is suggested by an analogy which will come before us presently.

(5.) *Any two Homographic Plane Figures.*

Since any two homographic plane figures, such that the lines at an infinite distance in the two figures are not corresponding lines, (this limitation is to be always understood in what follows when we speak of two homographic plane figures,) can be placed in perspective with one another, it appears that in any two such homographic systems there exist two pair of corresponding foci, and two pair of equi-segmental axes. This we shall now show independently of all considerations of perspective. Let P_1, P_2, q_1, q_2 be the imaginary circular points at an infinite distance in the planes Ω and ω respectively; and to P_1P_2, q_1q_2, let p_1p_2, Q_1Q_2 correspond in the planes ω and Ω respectively. The lines P_1P_2, p_1p_2; Q_1Q_2, q_1q_2, will be pairs of corresponding lines; P_1P_2, q_1q_2 being the lines at an infinite distance in the two planes, and p_1p_2, Q_1Q_2 the vanishing lines. Further, the three diagonal points of the quadrangle $P_1P_2Q_1Q_2$ (which are all real) will correspond to the three diagonal points of the quadrangle $p_1p_2q_1q_2$; of these three pairs of corresponding points, one pair are the points Y, y at an infinite distance on the vanishing lines; the two other pairs are the two pairs of foci. For if F, f are corresponding diagonal points (other than Y, y) of the two imaginary quadrangles, the homographic pencils at F, f are equiangular, because the imaginary circular asymptotes FP_1, FP_2 correspond in the pencil at F to the imaginary lines fp_1, fp_2, or fq_1, fq_2, *i.e.* to the imaginary circular asymptotes in the pencil at f. To determine the two pair of foci in two given homographic planes Ω and ω, we consider a pair of rectangular points at an infinite distance in each plane; let A_1A_2, b_1b_2 be these pairs of points; a_1a_2, B_1B_2 the pairs of points corresponding to them. The lines B_1B_2, a_1a_2 are the vanishing lines of the two planes; the centre of either plane is the point corresponding to the point at an infinite distance in the direction perpendicular to that of the vanishing line in the other plane; the focal axes are the lines perpendicular to the vanishing lines of the two planes at their respective centres; lastly, the foci are the points of intersection of the

focal axes by the circles described on B_1B_2, a_1a_2 as diameters, and are situated in each plane symmetrically with regard to its vanishing line. If, assuming that we view each plane from a determinate region in space, we regard the rotations round F_1, f_1 as similar, it is evident that the rotations round F_2, f_2 must be dissimilar, and *vice versâ*; otherwise the two homographic figures would be similar, and the lines at an infinite distance would be corresponding lines, contrary to the hypothesis. We might prove the same thing, by imagining the planes of the two figures to coincide. The circular asymptotes at F_1, f_1, and again at F_2, f_2 will then be corresponding lines. But the correspondence in one case will be direct, and in the other inverse (*i. e.*, in the one case, those asymptotes which run to the same imaginary circular point at an infinite distance will be corresponding lines, in the other case asymptotes running to opposite circular points will correspond). And, since the locus of the intersections of corresponding rays in two equiangular pencils is a circle, or an equilateral hyperbola, according as the rotations of the two pencils are in the same or in opposite directions, we infer that two equiangular pencils in the same plane have the same direction of rotation, or opposite directions, according as the circular asymptotes of the two pencils correspond directly or inversely.

As we have obtained the foci in each plane by a quadratic construction (which seems inevitable) we have still to determine their correspondence, and the corresponding directions of rotation round each. To do this, we have only to observe that each plane is divided by its focal and vanishing axes into four regions, which correspond to one another in a manner which is readily ascertained, because when we pass in either plane from one region A into another region B across one of their common boundaries, we must simultaneously pass in the other plane from the region corresponding to A into the region corresponding to B, and must traverse the corresponding boundary; (the line at an infinite distance is a common boundary, it will be observed, of two diametrically opposite regions). Thus we have only to ascertain in either plane the region which corresponds to a given region in the other; the correspondence of the remaining regions is then known, and with it the correspondence of the foci. Lastly, if O′, *o* be the points at an infinite distance on the focal axes of the two planes, the direction of rotation from FO′ to FY corresponds to the direction of rotation from *fo* to *fy*, the rotating radii vectores being supposed to move in corresponding regions in the two planes.

If A, *a* are corresponding points in the regions (Ω_2), (ω_2) respectively, the angles F_2F_1A, f_2f_1a, are both acute; they are, therefore, equal to one another, since, by virtue of the equiangularity of the pencils at F_1, f_1, they must be either supplementary or equal. Observing that the directions F_1F_2, F_1A are opposite in sign to the directions f_1f_2, f_1a, we see that

angles between corresponding directions on corresponding radii vectores are equal, in which form we have already stated the equiangular property of the foci.

The determination of the foci requires (as we have seen) the construction of three points in each plane corresponding to three points at an infinite distance in the other plane. And thus the actual determination of the foci, though very elementary in theory, is in actual practice somewhat troublesome. But when the foci have once been determined, the homographic representation of either plane upon the other can be carried out very rapidly; since, if A be any given point of Ω, F_1 the further, and F_2 the nearer focus, we have only to make the angle f_2f_1a equal to the angle F_2F_1A, and the angle f_1f_2a supplementary to the angle F_1F_2A; the directions in which these angles are to be measured being at once indicated by the correspondence of the regions in which A and a are situated.

The anharmonic equation $(O, X, F_1, \infty) = (\infty, x, f_1, o')$, or $OX \cdot o'x = -C \cdot c$, in which X, x denote corresponding points on the focal axes, suffices to prove that the parallels to the vanishing lines at Ω_1, ω_1; Ω_2, ω_2, are corresponding lines. And that these lines are equi-segmental, will then follow from the equiangular property of the foci, since $F_1\Omega_1 = f_1\omega_1$, $F_1\Omega_2 = -f_1\omega_2$. The image in either plane of any given indefinite line in the other is most easily found by making the intercepts on the equi-segmental lines in the first plane equal in sign and in magnitude to the corresponding intercepts in the second plane; so that, if the two intercepts are drawn in the same direction from the focal axis in one plane, they are drawn in opposite directions in the other plane.

(6.) *Homographic Plane Figures placed Homologically.*

It will be observed that any two homographic plane figures can be made homological, or put in plane perspective with one another, in four different ways. For we can take either pair of equi-segmental axes for the axis of homology, and either pair of foci for the centre of homology. And it is sometimes of importance, in the theory of homological figures, to consider the non-coincident foci and equi-segmental axes, as well as the two foci which are united in the centre of homology, and the two equi-segmental lines which are united in the axis of homology.

For example, if we regard a conic section as homological with itself, any point in the plane of the conic being the centre of homology, and its polar the axis of homology, the foot of the perpendicular let fall from the pole upon the polar will represent the second pair of foci (which in this case are coincident because the parameters are equal); and in like manner the second pair of equi-segmental lines will be represented by the parallel to the polar through the pole. Thus we have the elementary properties of a conic section, "angles subtended at the foot of the

perpendicular by chords passing through the pole are bisected by the polar;" "the pole is the point of bisection of intercepts on the parallel to the polar made by tangents at the extremities of chords passing through the pole," &c.

Again, if we regard two conics as homological, a point of intersection of their common tangents being the centre of homology, and the axis of homology being either of the two common chords which pass through the intersection of the polars of the centre of homology with regard to the two conics; it will be found that there is a second pair of foci, situated on the perpendicular let fall from the centre of homology upon the axis of homology, and that corresponding points of the two conics subtend equiangular pencils (with opposite rotations) at these two points. And, in like manner, corresponding lines in the two figures determine equal intercepts (but measured in opposite directions) upon two axes situated at the same distances from the second pair of foci, that the axis of homology is from the centre of homology.

(7.) *Homographic Plane Figures placed Symmetrically.*

The equality of the parameters of two plane homographic figures is the necessary and sufficient condition that they should be capable of being so placed in the same plane that each point shall have but one corresponding point. For if the two figures can be so placed as to have this symmetrical relation to one another, the imaginary points corresponding to the imaginary circular points at an infinite distance must coincide; and hence the imaginary chords Q_1Q_2, p_1p_2, and with them the real parameters, must be equal. Conversely, we can always render the two figures capable of a symmetric position by altering the linear dimensions of either of them in the ratio of its parameter to the parameter of the other; since after this alteration the imaginary chords Q_1Q_2, p_1p_2, will be equal, and can be made to coincide. There are then two positions of symmetry—viz., the two positions of homology in which the vanishing lines coincide. It is sometimes convenient to imagine the scale of one of the figures altered in the parametric ratio; we shall express this by saying that the figures are reduced to the same scale.

(8.) *Metrical Properties of the Focal Radii Vectores.*

The following elementary properties of the foci of two homographic plane figures are frequently useful:—

(*a*.) The focal radii vectores of two corresponding points P and p satisfy the equation

$$\frac{F_1P}{F_2P} + \frac{f_1p}{f_2p} = 0.$$

The truth of this equation, so far as absolute magnitude is concerned, appears immediately from a comparison of the triangles F_1PF_2, f_1pf_2; the two ratios are of opposite signs, because the radii vectores drawn from the nearer foci are of the same sign, and the radii vectores drawn from the further foci are of opposite signs.

(β.) The rectangle contained by the sum of the radii vectores of P, and the difference of the radii vectores of p, is equal to the rectangle contained by the difference of the radii vectores of P and the sum of the radii vectores of p; and either of these rectangles is equal to four times the rectangle of the parameters. This relation is expressed by the single equation

$$(F_1P+F_2P)\,(f_1p+f_2p) = F_1F_2 \cdot f_1f_2,$$

if we observe that in it the signs of F_1P and F_2P are arbitrary, and that the signs of f_1p and f_2p depend on the signs of F_1P and F_2P respectively. The truth of the equation may be inferred immediately from the elementary theorem, that if a straight line bisecting the angle of a triangle either internally or externally be produced to meet the base, the square of the bisecting line is equal to the rectangle contained by the sum of either side and the segment of the base adjacent to it, and by the difference of the other side and the segment adjacent to it.

(γ.) Let N, n be the feet of the perpendiculars let fall from P and p upon the vanishing lines. The distances NP, np are of opposite sign, and their rectangle (see Art. 5) is equal to the rectangle of the parameters. We then have the relations

$$\frac{F_1P}{NP} = \frac{f_1p}{o'f_1}, \quad \frac{F_2P}{NP} = \frac{f_2p}{o'f_2}, \quad \frac{f_1p}{np} = \frac{F_1P}{OF_1}, \quad \frac{f_2p}{np} = \frac{F_2P}{OF_2},$$

any one of which, combined with the equation $NP \cdot np = OF_1 \cdot o'f_1$, and with the equation (α), gives the other three. To prove them, we have only to observe that the quadrilaterals NPF_1O, $o'f_1pn$ are not only equiangular, but also similar, because $NP \cdot np = OF_1 \cdot o'f_1$.

If, in the construction of the point p, corresponding to a given point P, we wish to avoid the use of points lying on the further side of the vanishing line, we may either determine the ratio of f_1p to F_1P by one of the formulæ (γ); or, preferably, we may make the angle $f_1o'p$ equal to the angle OF_1N; the point p is then the intersection of $o'p$ and f_1p.

(δ.) If the two planes are in perspective, we have

$$\frac{F_1P}{f_1p} = \frac{SP}{Sp} \text{ for the similar foci,}$$

and

$$\frac{F_2P}{f_2p} = -\frac{SP}{Sp} \text{ for the dissimilar foci.}$$

(9.) *Circles changed into Circles.*

The circles of the system of which F_1, F_2 are the limiting points, and the vanishing line the radical axis, are transformed into circles of the system of which f_1, f_2 are the limiting points, and the vanishing line the radical axis; and these are the only circles in either figure which are changed into circles in the other figure. For it is evident that to the conics (P_1, P_2, Q_1, Q_2) there will correspond the conics (p_1, p_2, q_1, q_2); or again, if the locus of A be a circle of the system of which F_1, F_2 are the limiting points, the ratio $\frac{F_1A}{F_2A}$ is constant; therefore the ratio $\frac{f_1a}{f_2a}$ is also constant, (Art. 8, *a*); *i. e.*, the locus of *a* is a circle of the system of which f_1, f_2 are the limiting points. We shall, for brevity, call these two systems of circles the focal circles of the two planes.

It will be found that the radii of corresponding circles are to one another as the parameters; and that, if the figures be reduced to the same scale, and superposed so that their foci coincide, the corresponding circles will coincide (but not in respect of their corresponding points).

(10.) *The Homographic Modulus of Corresponding Pencils.*

In any two homographic pencils (A) and (*a*) there exists a pair of corresponding right angles (Steiner, "Systematische Entwickelung," p. 31); and if the pencils are not equiangular, there is only one such pair. We shall term these corresponding right-angles the right-angles of the pencils (A) and (*a*). Let A and *a* be any two corresponding points in the planes Ω and ω; the lines bisecting the angles F_1AF_2, f_1af_2, internally and externally, are the lines containing the right angles of the pencils at A and *a*. For the double rays of the pencil $A.[P_1P_2, Q_1Q_2, F_1F_2]$, (which is a pencil in involution, because P_1P_2, Q_1Q_2, F_1F_2 are the vertices of a quadrangle,) correspond to the double rays of the corresponding pencil $a.[p_1p_2, q_1q_2, f_1f_2]$. We might prove the same thing, without using imaginary points, by considering the corresponding circles which pass through A and *a*. And since the rectangle of the central abscissas of corresponding points is equal to the rectangle of the parameters, we see that the external bisector at either of the two points A or *a* answers to the internal bisector at the other.

If, in any two homographic pencils, Φ and ϕ are corresponding angles, measured from either pair of the corresponding rectangular lines, the ratio $\tan\Phi : \tan\phi$ is constant. This constant ratio we may term the homographic modulus of the two pencils. We observe (1) that the definition is applicable to homographic pencils in involution; (2) that the homographic modulus is positive or negative according as

corresponding directions of rotation in the two pencils are regarded as having the same sign or opposite signs; (3) that the homographic modulus of two equiangular pencils is $+1$ or -1; (4) that the definition is relative to a given pair of the corresponding rectangular lines, and that if for this pair we substitute the other pair, the homographic modulus changes into its reciprocal.

The homographic modulus of the pencils at A and a, taken relatively to the external bisector at A and the internal bisector at a, is evidently

$$\cot \tfrac{1}{2}\mathrm{A} \cot \tfrac{1}{2}a = \tan \tfrac{1}{2}(\mathrm{F}_1+\mathrm{F}_2) \tan \tfrac{1}{2}(f_1+f_2)$$
$$= \tan \tfrac{1}{2}(\mathrm{F}_1+\mathrm{F}_2) : \tan \tfrac{1}{2}(\mathrm{F}_1-\mathrm{F}_2) = \mathrm{R}_2+\mathrm{R}_1 : \mathrm{R}_2-\mathrm{R}_1 = r_2-r_1 : r_2+r_1,$$

the letters $\mathrm{A}, \mathrm{F}_1, \mathrm{F}_2$, a, f_1, f_2 denoting the internal angles of the triangles $\mathrm{F}_1\mathrm{AF}_2$, f_1af_2, and $\mathrm{R}_1, \mathrm{R}_2, r_1, r_2$ representing the absolute values of the focal radii vectores of A and a. We have supposed that F_1 is the nearer, F_2 the further focus, so that $\mathrm{F}_2 = f_2$, $\mathrm{F}_1+f_1 = \pi$.

(11.) *Angles changed into Equal or Supplementary Angles.*

In any two homographic pencils (A) and (a) there exists an infinite number of equal corresponding angles, and again an infinite number of supplementary corresponding angles (M. Chasles, "Géométrie Supérieure," Art. 147). We may add, that the angles in either pencil A, which are equal to their corresponding angles in the other pencil, form a pencil in involution, of which the right-angle is the right-angle of the pencil A, and of which the modulus is the homographic modulus of the two pencils taken positively; and, similarly, the angles in the pencil A which are supplementary to their corresponding angles form a pencil in involution which has the same right-angle as the pencil (A), and the same modulus, only taken negatively. The former involution always has real double lines, the latter never.

To find the involutions of equal angles at the corresponding points A and a; let F_1, f_1 be the nearer foci, let any circle passing through A and F cut the vanishing line in M_1, M_2, and let m_1, m_2 be the corresponding points at an infinite distance in the plane ω. The angle $\mathrm{M}_1\mathrm{AM}_2$ will be transformed into an equal angle m_1am_2; for the angles $\mathrm{M}_1\mathrm{AM}_2$, $\mathrm{M}_1\mathrm{F}_1\mathrm{M}_2$ are equal (not supplementary, since the chords $\mathrm{M}_1\mathrm{M}_2$, AF_1 do not intersect); *i. e.*, the angles $\mathrm{M}_1\mathrm{AM}_2$ and $m_1f_1m_2$, or finally $\mathrm{M}_1\mathrm{AM}_2$ and m_1am_2, are equal. It will be observed that the directions AM_1, am_1; AM_2, am_2 are corresponding directions.

To find the involutions of supplementary angles at the points A and a we have only, in the preceding construction, to substitute the further focus F_2 for the nearer focus F_1. The angles $\mathrm{M}^1\mathrm{AM}_2$, $\mathrm{M}_1\mathrm{F}_2\mathrm{M}_2$ will be supplementary (and not equal); so that $\mathrm{M}_1\mathrm{AM}_2$ will be transformed into a supplementary angle m_1am_2.

(12.) *Segments changed into Equal Segments.*

If H and h' are the *centres* of two homographically divided lines, (*i. e.* the points which on each line correspond to the points at an infinite distance upon the other line,) and if A, a are any two corresponding points on the two lines, the rectangle $\mathrm{H\,A} \times h'\,a$ (which we may term the rectangle of the homography) is constant.

In any two homographically divided lines there is an infinite number of segments equal to their corresponding segments and having the same sign, and again an infinite number of segments equal in absolute magnitude to their corresponding segments, but having opposite signs. Upon either line either set of segments form an involution of which the centre is the homographic centre of the line, and of which the rectangle is the rectangle of the homography, taken positively or negatively, according as the segments considered are equal to their corresponding segments with the same sign, or with opposite signs.

Thus, if 2D, $2d$ are the intercepts made by the cyclic axes on any two corresponding lines meeting the vanishing lines in H and h', the segments $\mathrm{X_1\,X_2}$ of the involution determined by the equation $\mathrm{HX_1} \times \mathrm{HX_2} = \mathrm{D} \times d$ are equal to the corresponding segments $x_1\,x_2$ of the involution $h'x_1 \times h'x_2 = \mathrm{D} \times d$; and, again, the corresponding segments of the involutions $\mathrm{HX_1} \times \mathrm{HX_2} = -\mathrm{D} \times d$, $h'x_1 \times h'\,x_2 = -\mathrm{D} \times d$ are equal, but have opposite signs.

And, in general, the segments of the involution $\mathrm{HX_1} \times \mathrm{HX_2} = \mu\mathrm{D} \times d$, where μ is any multiplier, are μ times the corresponding segments of the involution $\mu \times h'x_1 \times h'x_2 = \mathrm{D} \times d$. The segments of the two involutions are divided externally or internally by the vanishing lines, according as μ is positive or negative.

It thus appears that there are an infinite number of triangles in either plane, similar to their corresponding triangles, and having a given ratio to them. For take any point A in the plane Ω, any straight line passing through it, and any positive ratio; there is always one triangle (and only one, if A is not a focus, and the given line not a cyclic axis), having a vertex at A, and a side in the given straight line, which is transformed into an equiangular triangle of which the sides are to the sides of the triangle in the given positive ratio. The two corresponding triangles are not intersected by the vanishing lines, so that points interior to either triangle correspond to points interior to the other. If we do not attend to the sign of the given ratio, there are in all four triangles, each having a vertex at the point A, and a side upon the given line, which are equiangular to their corresponding triangles, and which have the given ratio for their ratio of similarity. For, if we do not attend to signs, there are two segments of the given line having a common extremity at A, which are in the given ratio to their corre-

sponding segments, and there are two lines passing through A which make angles with the given line equal to the corresponding angles. Of these triangles, that just considered is one; the other three are all intersected by the vanishing line; so that two of the sides of any one of them are to their corresponding lines in a negative ratio.

(13.) *The Confocal Conics.*

Every conic which has a focus at F_1 or F_2 is transformed into a conic having a focus at f_2 or f_1. For right-angles at F are transformed into right-angles at f, so that if the involution determined by the given conic at the point F be rectangular, the involution determined by the corresponding conic at the point f is also rectangular. And conversely, if a point in either figure, and its image in the other, be both foci of corresponding conics, the point and its image are corresponding foci in the two figures.

Two cases of this property are of special interest.

(i.) A circle having its centre at F is transformed into a conic, of which F is the focus, and the vanishing line the directrix. This follows, independently of the general property, from the equations (γ), which also show that the eccentricity of the conic is equal to the radius of the circle divided by the parameter; it thus varies directly as the radius of the circle.

(ii.) Conics in the plane Ω, of which F_1, F_2 are the foci, are transformed into conics of which f_1, f_2 are the foci, the ellipses into hyperbolas, and the hyperbolas into ellipses. We shall term these conics the confocal conics of the two homographic figures. An independent proof of the theorem is supplied by the formula (β); and conversely, the theorem may be used to establish that formula, since the rectangle contained by the major semi-axes of two corresponding confocal conics is evidently equal to the rectangle of the parameters.

The eccentricity of any confocal in either figure is the reciprocal of the eccentricity of the corresponding confocal, the asymptotes of the hyperbola containing the same angle as the focal radii vectores of the extremities of the minor axis of the ellipse. If the figures be reduced to the same scale, and the foci be superposed, corresponding confocals will intersect on the cyclic axes, and will thus have the same latus rectum.

If Σ and σ are any two corresponding confocals, the normals of Σ and the normals of σ are corresponding lines (Art. 10). Hence also the centre of curvature at any point of Σ corresponds to the centre of curvature at the corresponding point of σ; and the evolute of either is transformed into the evolute of the other. Again, any two *similar* arcs of Σ (*i. e.* arcs of which the difference is geometrically rectifiable) have for their corresponding arcs two similar arcs of σ; and the rectifying

tangents in either figure (*i. e.*, the tangents of which the difference is equal to the difference of the arcs) are images of the rectifying tangents in the other figure. In the same way, the polygon of a given number of sides, and of minimum perimeter, inscribed in any arc of a confocal conic of either figure, corresponds to the polygon of the same number of sides, and of minimum perimeter, inscribed in the corresponding confocal arc. It is hardly necessary to observe, that the elliptic integrals which express the lengths of corresponding arcs of corresponding confocal curves are not themselves equal to one another, and are not transformed into one another by the homographic transformation.

(14.) *The Indicatrix, or Strain Ellipse.*

The *indicatrix* at any point A of the plane Ω is the evanescent ellipse which is the image of an evanescent circle, having its centre at the corresponding point a. The indicatrix is, in fact, the "strain ellipse," if we regard any part of the plane Ω as a deformation of the corresponding part of the plane ω, produced by a mechanical strain. It is readily seen that, if we consider the radius of the evanescent circle at a as an infinitesimal of the first order, the distance of the centre of the strain ellipse from A will be an infinitesimal of the second order. For the determination in species of the strain ellipse at the point A, we have the theorem: "The strain ellipse is similar and similarly situated to the ellipse, of which the principal axes are normal, at the point A, to the confocal ellipse and hyperbola intersecting at that point, and are respectively equal to the major axes of those curves."

This auxiliary ellipse is no other than the ellipse employed by M. Chasles, in his solution of the problem, "To determine the principal axes of an ellipse, of which one pair of conjugate diameters are given in magnitude and position" (Aperçu historique des Méthodes en Géométrie, Note 25). M. Chasles has shown that a reciprocal relation subsists between the auxiliary ellipse and the ellipse of the confocal system which passes through its centre. Thus the centre of either ellipse lies on the circumference of the other; the major axis of either is normal to the other; the asymptotes of either pass through the imaginary foci of the other; the major and minor axis of either are respectively equal to the sum and difference of the focal radii vectores of its centre considered as a point on the circumference of the other; lastly, the distance between the real foci of either is equal to that diameter of the other which is conjugate to the diameter passing through the two centres.

To prove that the strain ellipse is similar and similarly situated to the auxiliary ellipse, it is sufficient to observe, that the asymptotes of the evanescent circle at a are aq_1, aq_2; and that, consequently, the strain

ellipse touches the imaginary lines AQ_1, AQ_2 at the points Q_1, Q_2. But the strain ellipse is infinitesimal; the imaginary lines AQ_1, AQ_2 are therefore its asymptotes, *i.e.*, it is similar and similarly situated to the auxiliary ellipse.

To determine, then, the strain ellipse in species, we have only to draw the focal radii vectores of the point A, and to bisect the angle contained by them internally and externally; the major and minor axes of the strain ellipse are respectively in the directions of the bisecting lines, and are proportional to the sum and difference of the radii vectores. It will be seen that the confocal hyperbolas are lines of greatest elongation (or least compression), and that the confocal ellipses are lines of least elongation (or greatest compression). The focal circles are lines of *similar distortion*, because for all points on any one of them the ratio of the two radii vectores is constant, and therefore the ratio of their sum and difference; *i. e.*, all points on the same focal circle have similar indicatrices.

It remains to find the absolute dimensions of the indicatrix. Let $d\Sigma_1$, $d\Sigma_2$ represent the elements of the arcs of the confocal hyperbola and ellipse which intersect at A; let $d\sigma_1$, $d\sigma_2$ represent the corresponding elements in the plane ω; and let Λ_1, Λ_2, λ_1, λ_2 be the semiaxes major of the curves Σ_1, Σ_2, σ_1, σ_2; so that, supposing F_1 the nearer focus, we have $\Lambda_2 = \frac{1}{2}(R_2+R_1)$, $\lambda_2 = \frac{1}{2}(r_2-r_1)$, $\Lambda_1 = \frac{1}{2}(R_2-R_1)$, $\lambda_1 = \frac{1}{2}(r_2+r_1)$, $\Lambda_1\lambda_1 = \Lambda_2\lambda_2 = Cc$. Considering two points on Σ_1 and Σ_2 indefinitely near to A, and denoting, as in Art. 10, the angles F_1AF_2, f_1af_2 by A and a, we find

$$d\Sigma_1 = \frac{d\Lambda_2}{\cos\frac{1}{2}A}, \qquad d\sigma_1 = -\frac{d\lambda_2}{\sin\frac{1}{2}a},$$

$$d\Sigma_2 = -\frac{d\Lambda_1}{\sin\frac{1}{2}A}, \qquad d\sigma_2 = \frac{d\lambda_1}{\cos\frac{1}{2}a}.$$

But
$$\frac{d\Lambda_1}{\Lambda_1} + \frac{d\lambda_1}{\lambda_1} = 0, \qquad \frac{d\Lambda_2}{\Lambda_2} + \frac{d\lambda_2}{\lambda_2} = 0;$$

so that
$$d\Sigma_1 = \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}A} \cdot \frac{1}{\lambda_2} \cdot \Lambda_2\, d\sigma_1,$$

$$d\Sigma_2 = \frac{\cos\frac{1}{2}a}{\sin\frac{1}{2}A} \cdot \frac{1}{\lambda_1} \cdot \Lambda_1\, d\sigma_2.$$

Again, substituting for $\frac{1}{2}a$ its value $\frac{1}{2}(F_1-F_2)$, we find, from the triangles F_1AF_2, f_1af_2,

$$\frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}A} = \frac{\Lambda_1}{C}, \qquad \frac{\cos\frac{1}{2}a}{\sin\frac{1}{2}A} = \frac{\Lambda_2}{C};$$

whence, if i be the radius of the evanescent circle at a, the major and minor semi-axes of the strain ellipse at A are respectively

$$\text{(A)} \ldots\ldots \frac{\Lambda_1\Lambda_2}{Cc} \cdot \frac{\Lambda_2}{C}\, i, \qquad \frac{\Lambda_1\Lambda_2}{Cc} \cdot \frac{\Lambda_1}{C}\, i.$$

From these expressions it follows that, given in magnitude the parameters of the two planes, and given in position a single pair of corresponding points A and a; given also in position and magnitude the indicatrix at one of these points, for example at A; the homography of the two planes is determined. For the coefficients,

$$\frac{\Lambda_1\Lambda_2}{Cc} \times \frac{\Lambda_2}{C}, \qquad \frac{\Lambda_1\Lambda_2}{Cc} \times \frac{\Lambda_1}{C},$$

being given, the values of Λ_1 and Λ_2 may be found (by the extraction of a cube root): thus the auxiliary ellipse at A is completely determined. The vanishing line of the plane Ω is one of the four tangents of the auxiliary ellipse which are parallel to a diameter equal to 2C, and the homographic centre is the point of contact. Similarly may the vanishing line and centre be determined in the plane ω; since the indicatrix at a can be found when that at A is given.

(15.) *The Canonical and Elliptic Equations of a Plane Homography.*

If X, Y, x, y are the coordinates of two corresponding points, the focal axis and vanishing line in each plane being taken as the axes of coordinates, we have

$$Xx = Cc,$$

$$\frac{Y}{X+C} = \frac{y}{x+c},$$

the former equation being equivalent to the anharmonic equation of Art. 5, the latter expressing the property of the corresponding foci of which the abscissæ are $-C$ and $-c$. These equations, written in either of the forms

$$\text{(B)} \ldots\ldots \begin{cases} Xx = Cc \\ Yx = Cy \end{cases} \quad \begin{cases} Xx = Cc \\ Xy = cY \end{cases}$$

may be termed the canonical equations of the homography (M. Chasles, "Géométrie Supérieure," Art. 533), and may be employed to verify analytically the preceding results. It will be remembered that the axes of y and Y are not corresponding lines, neither are the origins corresponding points. Thus the abscissæ of corresponding points are not corresponding lines, and indeed are not measured in corresponding directions; but the ordinates of corresponding points (considered as lines drawn from the extremities of the abscissæ parallel to the vanishing lines) are corresponding lines.

The elliptic coordinates Λ_1, Λ_2; λ_1, λ_2 of two corresponding points (*i. e.*, the major semi-axes of the confocal conics passing through the points) are, as we have seen, connected by the relations

$$\text{(B')} \ldots\ldots \Lambda_1\lambda_1 = \Lambda_2\lambda_2 = Cc.$$

Thus every homographic transformation of a plane figure, in which the line at an infinite distance is transformed into a line at a finite distance, is equivalent to an inverse transformation of the elliptic coordinates of the points of the plane. In this way the expressions already given for the semi-axes of the indicatrix may be immediately deduced from the elementary elliptic formulæ

$$d\Sigma_1 = \sqrt{\left(\frac{\Lambda_2^2 - \Lambda_1^2}{\Lambda_2^2 - C^2}\right)}\, d\Lambda_2, \quad d\Sigma_2 = -\sqrt{\left(\frac{\Lambda_2^2 - \Lambda_1^2}{C^2 - \Lambda_1^2}\right)}\, d\Lambda_1,$$

combined with the corresponding formulæ for the plane ω. Again, using the formulæ $\Lambda_1\Lambda_2 = CX$, $\lambda_1\lambda_2 = cx$, we may write those expressions in either of the forms

$$(A) \ldots\ldots\ldots\ldots \frac{X}{\lambda_2} \cdot i, \quad \frac{X}{\lambda_1} \cdot i; \quad \text{or} \quad \frac{\Lambda_2}{x} \cdot i, \quad \frac{\Lambda_1}{x} \cdot i.$$

Thus the ratio of corresponding elementary areas at A and a is that of X^2 to $\lambda_1\lambda_2$, or of $\Lambda_1\Lambda_2$ to x^2, or of $C^{\frac{1}{2}}X^{\frac{3}{2}}$ to $c^{\frac{1}{2}}x^{\frac{3}{2}}$; *i. e.*, it varies in the sesquiplicate ratio of the distances of the two areas from the vanishing lines. The lines $X^3 = \pm Cc^2$, $x^3 = \pm C^2c$, (of which two in each plane are real, and four imaginary,) are lines at which corresponding elementary areas are equal. More generally, the lines $kX = \pm KC$, $Kx = \pm kc$ are the real lines, in the planes Ω and ω, at which corresponding evanescent areas are to one another in the ratio of K^3C^2 to k^3c^2.

(16.) *Theorems relating to Curvature.*

Since evanescent segments, at the same point, and upon the same straight line, are altered in one and the same ratio in any homographic transformation, the curvature of all curves which touch one another at a given point is altered in one and the same ratio. Thus, if a curve touch a focal circle of the plane Ω, its radius of curvature at the point of contact is altered in the transformation in the ratio of C to c. Again, it will be found that the radius of curvature of a curve at a point at which its tangent is parallel to the vanishing line is altered in the same parametric ratio. Hence if we consider in the plane Ω any conic which passes through the imaginary points Q_1, Q_2 (and which, consequently, is transformed into a circle), it has the same curvature at the two points where it is touched by focal circles, and at the two points where it is touched by parallels to the vanishing line; for the radius of curvature at any one of these four points is to the radius of the corresponding circle in the ratio of the parameters. We thus obtain incidentally a solution of the problem, "Given a system of circles, and a conic, having the same radical axis, to determine the two circles of the system which touch the conic;" for the points of contact are at the extremities of the diameter equal to the diameter conjugate to the radical axis. In par-

ticular, the radius of curvature of the indicatrix at the points where its tangent is parallel to the vanishing line, or to the tangent of the focal circle passing through its centre, is to the radius of the corresponding evanescent circle in the ratio of the parameters: thus, if R is the radius of curvature of the auxiliary ellipse at the point O, Ai, Bi the principal semi-axes of the indicatrix, we have the equations

$$\frac{Ai}{\Lambda_2}R = \frac{C}{c}i, \quad \frac{Bi}{\Lambda_1}R = \frac{C}{c}i,$$

which are in accordance with the equations (A), since $R = \frac{C^3}{\Lambda_1\Lambda_2}$. More generally, if D$i$ is any semi-diameter of the indicatrix at the point A, the radius of curvature of any curve touching that semi-diameter at the point A is altered in the ratio of D^3 to AB (since $\frac{D^3}{AB}i$ is the radius of curvature of the indicatrix at the extremities of the diameter conjugate to Di). It will be seen, that of all curves passing through the point A, those which touch the confocal hyperbola at A experience in the transformation the greatest augmentation (or the least diminution) of curvature, and those which touch the confocal ellipse experience the greatest diminution (or the least augmentation) of curvature; so that the confocal conics may be said to be loci of greatest and least augmentation (or diminution) of curvature. The ratio of the radius of curvature of any curve passing through the point A to the radius of curvature of the corresponding curve is thus intermediate between the ratios $\Lambda_2^3 : C^2c$ and $\Lambda_1^3 : C^2c$. Let $K^3 : C^2c$ be any ratio intermediate between these two; there are evidently two equal semi-diameters of the indicatrix at the point A such that the radii of curvature of curves touching either of them are altered in the ratio $K^3 : C^2c$. If Φ be the angle made by either of these semi-diameters with the major axis of the indicatrix, the equation $\frac{D^3i}{AB} = \frac{K^3}{C^2c}i$ becomes, on substituting for A and B their values given by the equations (A),

$$(C) \ldots\ldots\ldots \frac{\cos^2\Phi}{\Lambda_2^2} + \frac{\sin^2\Phi}{\Lambda_1^2} = \frac{1}{K^2},$$

or

$$(C') \ldots\ldots \lambda_2^2\cos^2\Phi + \lambda_1^2\sin^2\Phi = \frac{C^2c^2}{K^2}.$$

From the first of these equations we learn that the two semi-diameters coincide in direction with those semi-diameters of the auxiliary ellipse which are equal to K; the second implies that the angles made by the two semi-diameters with the axis major of the indicatrix at A are equal to the angles made in the plane ω with the major axis of the indicatrix at a by the tangents drawn from the point a to that confocal conic which, in the plane ω, corresponds to the confocal conic of semi-axis major K in the plane Ω.

(17.) *Curves of Constant Alteration of Curvature.*

We may also regard the equation (C) as equivalent to the differential equation of a system of curves such that at any point on any one of them its radius of curvature is altered in the constant ratio of K^3 to C^2c. Substituting for $\tan^2\Phi$ its value,

$$-\frac{\Lambda_2^2-C^2}{\Lambda_1^2-C^2}\cdot\frac{d\Lambda_1^2}{d\Lambda_2^2},$$

we find for this differential equation the expression

$$\frac{d\Lambda_2}{\Lambda_2}\sqrt{\left(\frac{\Lambda_2^2-K^2}{\Lambda_2^2-C^2}\right)}=\pm\frac{d\Lambda_1}{\Lambda_1}\sqrt{\left(\frac{K^2-\Lambda_1^2}{C^2-\Lambda_1^2}\right)}.$$

The integral of this equation is easily obtained in a finite form; it seems, however, too cumbrous for discussion. It contains an algebraical function raised to the power $\frac{K}{C}$, but no other transcendental function. If, therefore, $\frac{K}{C}$ be rational, *i. e.*, if the given ratio $K^3 : C^2c$ is a multiple of the parametric ratio by the cube of a rational number, the curves of constant alteration of curvature are algebraic and of finite dimensions; in every other case they are transcendental. They lie entirely outside the confocal conic (K), and seem to meet it in cusps of which the tangents are normal to it. If $K=C$, *i. e.*, if the ratio is the parametric ratio, the curves of constant alteration of curvature are given by the equations

$$\frac{d\Lambda_2}{\Lambda_2}=\pm\frac{d\Lambda_1}{\Lambda_1},$$

or

$$\frac{\Lambda_1}{\Lambda_2}=\text{constant},\quad \Lambda_1\Lambda_2=\text{constant};$$

they are thus the focal circles and the parallels to the vanishing line, as we have already seen.

The orthogonal trajectories of the curves of constant alteration of curvature are always algebraic; they have for their differential equation

$$\frac{\Lambda_2 d\Lambda_2}{\sqrt{(\Lambda_2^2-C^2)(\Lambda_2^2-K^2)}}=\frac{\Lambda_1 d\Lambda_1}{\sqrt{(C^2-\Lambda_1^2)(K^2-\Lambda_1^2)}}.$$

(18.) *Curves of Constant Elongation.*

Let $\theta C : c$ represent any given ratio; there are in general two equal semi-diameters of the indicatrix at any point A which are in that ratio to the radius i of the corresponding evanescent circle. If, however, $\theta\frac{C}{c}i$ be greater than the major axis, or less than the minor axis of the indicatrix, the two diameters are imaginary. The equations

$$\Lambda_2^2\Lambda_1 = \theta C^3 \ldots\ldots\ldots (\alpha), \qquad \Lambda_1^2\Lambda_2 = \theta C^3 \ldots\ldots\ldots (\beta),$$

represent two loci, which separate the parts of the plane Ω, in which the two diameters are real, from those parts in which they are imaginary; at points on the locus (α) the two diameters coincide with the axis major of the indicatrix; at points on the locus (β) they coincide with its axis minor. The two loci are included in the same Cartesian equation of the sixth order

$$\theta^2C^2X^2Y^2 = (X^2-\theta^2C^2)\,(X^2-\theta C^2)\,(X^2+\theta C^2),$$

which represents a curve symmetrical with respect to the vanishing line and focal axis, and having a quadruple point at Y (the point at an infinite distance on the vanishing line). The two branches, one on each side of the vanishing axis, which touch it and one another at the point Y, form the locus (α); the locus (β) consists of the two branches which have the line at an infinite distance for their common tangent at the point Y. In the space included between the two branches of the locus (α), the semi-axis major of the indicatrix is less than $\theta\dfrac{C}{c}i$; in the spaces intermediate between the loci (α) and (β) the two diameters are real, and they are again imaginary in the spaces interior to the locus (β). To determine the angle Φ, which either of the two diameters makes with the axis major of the indicatrix, we observe that these lines coincide in direction with the two diameters of the auxiliary ellipse which are equal to $2\theta\dfrac{C^3}{\Lambda_1\Lambda_2}$; we thus obtain the equation

$$(D)\ldots\ldots\ldots \Lambda_1^2\cos^2\Phi + \Lambda_2^2\sin^2\Phi = \frac{\Lambda_1^4\Lambda_2^4}{\theta^2C^6} = \frac{X^4}{\theta^2C^2},$$

which implies that the two diameters coincide in direction with the tangents drawn from the point A to the confocal conic of which the semi-axis major is $\dfrac{X^2}{\theta C}$. Substituting for $\tan^2\Phi$ its value, we find

$$\frac{\Lambda_2^4\Lambda_1^2 - \theta^2C^6}{\Lambda_2^2 - C^2}\cdot\frac{d\Lambda_2^2}{\Lambda_2^2} = \frac{\Lambda_1^4\Lambda_2^2 - \theta^2C^6}{\Lambda_1^2 - C^2}\cdot\frac{d\Lambda_1^2}{\Lambda_1^2},$$

which is the differential equation of the curves of constant elongation, and seems not to admit of integration in any finite form. Its equivalent in Cartesian coordinates is

$$\frac{dX^2+dY^2}{C^2} = \theta^2\frac{dx^2+dy^2}{c^2},$$

or

$$\frac{X^4(dX^2+dY^2)}{\theta^2C^2} = C^2dX^2 + (YdX - XdY)^2.$$

It will be observed that $\Lambda_1\Lambda_2 = \theta C^2$, or $X = \theta C$, is a particular integral of the equation.

The following is an important property of the curves of constant elongation:—

"The intercept in the plane Ω on any tangent to one of these curves between the point of contact and the vanishing line is in the given ratio to the similarly defined segment on the corresponding tangent in the plane ω."

To establish this property, we have only to observe that the point of contact is one of the double points of that involution upon the tangent of which the segments are to their corresponding segments in the given ratio. Or we may infer it from the equation (D), with the help of the easily demonstrated theorem—

"If Ψ and ψ are the angles made with the vanishing line by any two corresponding tangents to the confocals (Γ) and (γ), the ratio $\frac{\sin\Psi}{\sin\psi}$ is constant and equal to $\frac{\Gamma}{C}$ or $\frac{c}{\gamma}$."

(19.) *Curves of Equal Tangential Deflexion.*

Through any point A of the plane Ω there pass two curves, such that the angle between any two tangents to either of them is equal to the angle between the two corresponding tangents of the corresponding curve. We may term these curves the curves of equal tangential deflexion. If Di be the length of the semi-diameter of the indicatrix, which touches one of these curves at the point A, and if $d\Theta = d\theta$ be the angle contained between this semi-diameter and a consecutive tangent to the curve, we find, since all areas at the point A are altered in the same ratio,

$$D^2 i^2 d\Theta : i^2 d\theta :: AB i^2 : i^2, \quad \text{or simply} \quad D^2 = AB.$$

Denoting by Φ the angle made by the semi-diameter Di with the major axis of the indicatrix, and substituting for A, B, and D, their values, we have for the differential equation of the curves of equal tangential deflexion

$$(E) \ldots\ldots \frac{\cos^2\Phi}{\Lambda_2} = \frac{\sin^2\Phi}{\Lambda_1},$$

or

$$\frac{d\Lambda_1}{\sqrt{\Lambda_1}\sqrt{(C^2-\Lambda_1^2)}} = \pm \frac{d\Lambda_2}{\sqrt{\Lambda_2}\sqrt{(\Lambda_2^2 - C^2)}}.$$

These curves are always algebraic; for, putting $\Lambda_1 = Cu_1^2$, $\Lambda_2 = \frac{C}{u_2^2}$, we have

$$\frac{du_1}{\sqrt{(1-u_1^4)}} = \pm \frac{du_2}{\sqrt{(1-u_2^4)}},$$

of which the integral is algebraic.

The equation (E) may be also obtained by observing that the curves of equal tangential deflexion which pass through the point A, must touch at that point the double lines of the pencil in involution, which

is equiangular to the corresponding pencil, and that the equation (E) is the equation determining these double lines (see Arts. 10 and 11). It appears from this that the angle at which the two curves intersect is always supplementary to the corresponding angle.

(20.) *Curves similar and similarly situated to their images.*

As an additional example of the use of the formulæ (B), let us propose to determine the conic sections, which in either figure are transformed into conics similar and similarly situated with regard to the vanishing line and focal axis of the other figure. If

$$aX^2+a'Y^2+a''C^2+2bCY+2b'CX+2b''XY=0$$

be the equation of any conic in the plane Ω, the equation of the corresponding conic is

$$a''x^2+a'y^2+ac^2+2b''cy+2b'cx+2bxy=0.$$

And if these two conics are similar and similarly situated, we must have $a=a''$, $b=\pm b''$; *i. e.*, every conic for which one of the two foci is the pole of the line parallel to the vanishing axis, and passing through the other focus, (or, which is the same thing, any conic for which one of the pairs of lines joining its points at an infinite distance to its points on the vanishing axis, intersect at a focus,) is transformed into a similar and similarly situated conic, the ratio of similarity being that of the parameters. There are thus two sets of conics (each forming a triply indeterminate linear system) which satisfy the conditions of the problem; but the conics of only one set at a time can be regarded as similarly situated to the corresponding conics, because in determining the two sets different directions on the vanishing lines are taken to determine the similarity of position.

Again, the corresponding conics will have their areas in the ratio of the squares of the parameters, if $(aa'-b''^2)^3=(a'a''-b^2)^3$; *i. e.*, the areas of all conics with regard to which the lines parallel to the vanishing line and passing through the foci are self-conjugate lines are to the areas of their corresponding conics in the duplicate ratio of the parameters. The only real conics of which the area is changed in this ratio are those defined by this geometrical condition; they form a quadruply indeterminate linear tangential system. But the analytical condition is also satisfied by the imaginary conics in the plane Ω, with regard to which the imaginary lines $X=\pm\rho C$, or $X=\pm\rho^2 C$, are harmonically conjugate, ρ denoting an imaginary cube root of unity. More generally, it will be found that the conics of which the area is changed in any given ratio are those which have for a pair of conjugate lines the two straight lines at which elementary areas are changed in the given ratio (Art. 15). If the corresponding conics are hyperbolas, we may substitute for the area in this result the triangle contained by the asymptotes and any tangent.

Lastly, the geometrical condition that a conic in either plane should be similar to its corresponding conic is that the pairs of points in which it intersects the vanishing line and the line at an infinite distance should subtend equal angles at a focus. But the quadruply indeterminate system determined by this condition is not a linear one.

Theorems of a similar kind to the preceding, but relating to curves of a higher order, may be obtained by observing that symmetrical functions of X, x; Y, y; or again of Λ_1, λ_1; Λ_2, λ_2 are unchanged by the transformation. Thus any curve represented by

$$F\left(X+\frac{Cc}{X},\ \frac{cY^2}{X},\ \ Y+\frac{cY}{X}\right)=0,$$

or again by

$$f\left(\Lambda_1+\frac{Cc}{\Lambda_1},\ \ \Lambda_2+\frac{Cc}{\Lambda_2}\right)=0,$$

is transformed into a curve similar and similarly situated with regard to the focal axis.

B.—Focal Properties of Two Homographic Point-Figures.

(21.) *The Imaginary Cones corresponding to Evanescent Spheres.**

By a point-figure we shall here understand a system of straight lines and planes passing through a point which is termed the centre of the point-figure. Let S, s be the centres of two point-figures, homographically related to one another; and let P, q represent the evanescent spheres (here to be regarded as imaginary cones), which have their centres at S, s. Excluding altogether from consideration the very particular case in which these two imaginary cones correspond to one another homographically, and in which, consequently, the two figures admit of exact coincidence with one another, let us represent by p, Q the imaginary cones, which in the figures s, S correspond to the cones P, q. We observe that if either p or Q is a cone of revolution, the other is so too; for if the cones P, Q have double contact, so also have the corresponding cones p, q. We shall hereafter (Art. 38) return for a moment to this particular case, but for the present we shall suppose that neither p nor Q is a cone of revolution.

(22.) *The Principal Axes.*

On this supposition there exists in each pencil one, and only one, system of straight lines at right angles to one another, such that their corresponding lines are also at right angles to one another. These lines are the principal axes of the cones Q and p. For the principal axes of Q are the system of self-conjugate axes common to the cones P and Q; these principal axes, therefore, correspond to the system of axes self-conjugate with regard to p and q; *i. e.* to the principal axes

* See Note at end of Paper, p. 248.

of p. We shall call these two sets of rectangular axes the principal axes of the two figures, and we shall distinguish them as the axes of XYZ, *xyz*.

(23.) *The Focal Lines and Cyclic Planes.*

To the four imaginary lines of intersection of P and Q, and to the four imaginary tangent planes common to those two cones, there correspond the four lines of intersection, and the four common tangent planes, of p and q. Hence to C_1 and C_2, the two real cyclic planes of Q, and to F_1, F_2, the two real focal lines of Q, there correspond c_1, c_2, the cyclic planes, and f_1, f_2, the focal lines of p. It will be observed that the real focal lines of an imaginary cone (differing in this respect from the focal lines of a real cone) lie in that principal plane of the cone to which the cyclic planes are perpendicular. We shall call the axis, in which the cyclic planes intersect, and which is perpendicular to the focal plane (*i. e.* to the plane containing the focal lines), the mean axis; of the two axes in the focal plane, we shall term that the major axis which makes with either cyclic plane, and with either focal line, acute angles together less than a right angle.

(24.) *The Reciprocity of the Imaginary Cones.*

The imaginary cones Q and p are reciprocal. Let Y, y be their mean axes; XZ, *xz*, their focal planes; P_1P_2, Q_1Q_2, and p_1p_2, q_1q_2, the imaginary lines in which these planes meet the cones P, Q and p, q respectively. From the anharmonic equation

$$(1).....[P_1, P_2, Q_1, Q_2, X, Z] = [p_1, p_2, q_1, q_2, x, z]$$

we infer the equation

$$[P_1, P_2, Q_1, Q_2] = [q_1, q_2, p_1, p_2],$$

which implies that the imaginary angles Q_1SQ_2, p_1sp_2 are equal, because P_1, P_2 and q_1, q_2 are pairs of lines representing evanescent circles. Again, X, Z and x, z are harmonic conjugates of the pairs P_1P_2, Q_1Q_2, and p_1p_2, q_1q_2 respectively. Hence we must have either the equation

$$[P_1, P_2, Q_1, Q_2, X, Z] = [q_1, q_2, p_1, p_2, x, z],$$

or else the equation

$$[P_1, P_2, Q_1, Q_2, X, Z] = [q_1, q_2, p_1, p_2, z, x].$$

But the former equation is inadmissible; for, on combining it with (1), we obtain

$$[p_1, q_1, x, z,] = [q_1, p_1, x, z],$$

which is untrue, since p_2, and not q_1, is the harmonic conjugate of p_1 with regard to *xz*. It is, therefore, the latter equation which subsists; it implies that Q_1 or Q_2 makes the same angles with X, that p_1 or p_2 makes with z; *i. e.*, that the angle Q_1SX is the complement of p_1sx. Similarly, the angle which the axis of X makes with either of the lines of Q which lie in the plane XY, is the complement of the corresponding angle in the plane *xy*; that is to say, the two cones are reciprocal.

It is evident that the mean axis of Q corresponds to the mean axis of p. Let C, c be the acute angles F_1SX, f_1sx; then the acute angle contained by YX and either cyclic plane of Q is the complement of c, and the acute angle contained by yx and either cyclic plane of p is the complement of C. Hence X is the major or minor axis of Q, according as $C < c$, or $C > c$; and to the major axis of Q the minor axis of p corresponds, and *vice versâ*. Neither of the angles C, c can be zero, nor a right angle, nor can they be equal to one another.

The reciprocity of the cones Q and p gives rise to a reciprocal relation between the two homographic figures, which may be thus stated. Conceive the two figures placed with their corresponding principal axes coincident. Let A, a be any two corresponding planes in the figures S and s; let b be the normal to A at the common centre of the figures; and B the normal to a at the same point; then B and b are corresponding lines in the figures S and s. If, therefore, we consider any two corresponding systems of planes and lines in S and s, the *reciprocal* systems of lines and planes will also be corresponding systems in s and S. Thus all the properties (metrical as well as descriptive) of two homographic point-figures are double, and we have an uniform method for passing from any property to its correlative.

(25.) *The Correspondence of Directions.*

The angles contained by planes intersecting in a focal line of S are equal to the corresponding angles contained by planes intersecting in a focal line of s; and, correlatively, the angles contained by lines intersecting at S in one of the cyclic planes of S, are equal to the corresponding angles in a cyclic plane of s. These theorems are evident, because the imaginary tangent planes of P, which intersect in F_1, correspond to the imaginary tangent planes of q, which intersect in f_1; and similarly, the lines in which P is intersected by either cyclic plane of Q, correspond to the lines in which q is intersected by either cyclic plane of p.

To fix the correspondence of the directions of rotation round either pair of corresponding focal lines, or in either pair of corresponding cyclic planes, we consider the intersections of the planes and lines of S and s by the surfaces of two spheres of radius unity having their centres at S and s. Let A, B, C be three points on the sphere S, forming a spherical triangle; it will be remembered that three points, not in the same great circle, always form one, and only one, spherical triangle, if by a spherical triangle we understand (as is usually done), a triangle formed by arcs of great circles, each of which is less than two right angles. As corresponding point to any point A on the sphere S, we might take either of two diametrically opposite points a, a' on the sphere s. But for one of these points (for example a) the corresponding directions of

rotation round A and a are similar (*i. e.*, both right handed or both left handed, when viewed from the centres of the spheres); while for the other point a' the corresponding directions of rotation are dissimilar. Let then a, b, c be the three points which on the sphere s correspond with similar rotations to the points A, B, C. These three points are thus determined without any ambiguity, and we shall now show that to points in the interior of the triangle ABC there correspond, with similar rotations, points in the interior of abc. The proof of this important theorem depends on the two principles: (i.), that if a point move continuously on either sphere, and traverse any curve on that sphere, its corresponding point on the other sphere simultaneously traverses the corresponding curve; (ii.), that if A and a are corresponding points with similar rotations, and if, while A moves continuously to B, a moves continuously to b, then a and b are also corresponding points with similar rotations. The first of these principles may be considered as evident; to establish the second, it will suffice to consider A and B as consecutive positions of A, so that while A describes the element AB, a describes the element ab. Let E be any great circle not intersecting AB, then the corresponding great circle e does not intersect ab, and if these two great circles be described by corresponding points V and v, the vector arcs AV, av will by hypothesis revolve in similar directions. But the arcs AV, BV evidently revolve in similar directions, and so do the arcs av, bv; *i. e.*, the corresponding rotations round B and b are similar. Let us now suppose that a point sets out from B, and describes the side BC of the triangle ABC; the corresponding point will at the same time describe the side bc of the triangle abc; for as it must not traverse either of the great circles ab, ac, it cannot describe an arc greater than a semicircle. Thus, to the points of any side of ABC there correspond, with similar rotations, the points of the corresponding side of abc. Let V be any point internal to ABC, let AV cut BC in A_1, and let a_1 on bc correspond to A_1 on BC; then AA_1B, aa_1c are corresponding spherical triangles, with similar rotations at their corresponding points; therefore the points of aa_1 correspond, with similar rotations, to the points of AA_1; *i. e.*, the point v, which corresponds with similar rotation to V, lies on aa_1 in the interior of the triangle abc.

The great circles which form the triangles ABC, abc, divide the spheres S and s each into eight spherical triangles, which correspond to one another one by one, with similar rotations at their corresponding vertices, just as the triangles ABC, abc. Thus each sphere is divided into eight regions, corresponding to the eight regions of the other sphere, in such a manner, that, if any point be taken on either sphere, the point which corresponds to it with similar rotation lies in the corresponding region of the other sphere.

We shall now take for ABC one of the eight octantal triangles XYZ,

and for abc the corresponding octant xyz; we shall denote by F_1, f_1; $Y\Omega_1$, $y\omega_1$, the foci and cyclic arcs which lie in the octants XYZ, xyz; and by F_2, f_2; $Y\Omega_2$, $y\omega_2$; the foci and cyclic arcs which lie in the octants $XY\bar{Z}$, $xy\bar{z}$; so that the directions of rotation round the corresponding foci F_1, f_1 and F_2, f_2, will be similar, and the directions $Y\Omega_1$, $y\omega_1$; $Y\Omega_2$, $y\omega_2$, will be corresponding directions on the cyclic arcs. It will be convenient to consider only the hemispheres of which the points X, x are the spheric centres, and the planes YZ, yz the bases. Thus, to any given point on the hemisphere S (not lying on the base circle itself), there corresponds one point, and only one, on the hemisphere s; and again any two great circles upon either hemisphere (neither of which is the base circle) intersect one another only in one point. To find the point a of the hemisphere s, which corresponds to a given point A of the hemisphere S, we draw the vector arcs F_1A, F_2A, and make the angles f_2f_1a, f_1f_2a equal in sign and magnitude to the angles F_2F_1A, F_1F_2A; the point of intersection of the arcs f_1a, f_2a is the point a required. Similarly, to find the great circle a of the hemisphere s, which corresponds to a given great circle A of the hemisphere S, we find the points D_1, D_2, in which A intersects the cyclic arcs of S, and we make the arcs ω_1d_1, ω_2d_2, equal in sign and magnitude to the arcs Ω_1D_1, Ω_2D_2; the arc d_1d_2 is the arc required.

(26.) *The Confocal Spherical Conics.*

The spherical conics of which F_1, F_2 are the foci are transformed into the spherical conics of which f_1, f_2 are the foci. This is evident from the equiangular property of the foci; or, again, if R_1, R_2, r_1, r_2 are the focal radii vectores of the corresponding points Δ and δ, the spherical triangles $F_1\Delta F_2$, $f_1\delta f_2$ give the equations

$$\frac{\tan\frac{1}{2}(R_1+R_2)}{\tan\frac{1}{2}(r_1+r_2)} = \frac{\tan\frac{1}{2}(R_1-R_2)}{\tan\frac{1}{2}(r_1-r_2)} = \frac{\tan C}{\tan c},$$

which imply that if $R_1 \pm R_2$ is constant, $r_1 \pm r_2$ is also constant. Thus the ellipses are transformed into ellipses, and the hyperbolas into hyperbolas, these denominations being relative to the two foci lying on each of the hemispheres S and s. If $\Lambda = \frac{1}{2}(R_1+R_2)$, or $= \frac{1}{2}(R_1-R_2)$, is the focal semi-axis of one of the confocal conics of S, the quotient $\frac{\tan\Lambda}{\tan C}$, which is one of the spherical eccentricities of the conic, remains unchanged in the transformation; for, if λ be the semi-axis of the corresponding conic, we have the equation

$$\frac{\tan\Lambda}{\tan\lambda} = \frac{\tan C}{\tan c},$$

which results from the homography of corresponding points of the

great circles XZ, xz. It is also evident that if we consider any two corresponding conics of the two confocal systems, there will correspond to one another in the two figures—the normal arcs of the two curves, their spherical centres of curvature, their evolutes, their *similar* arcs, as also the spherical polygons of minimum perimeter circumscribing corresponding arcs, and the spherical polygons of maximum perimeter inscribed in corresponding arcs.

(27.) *The Concyclic Spherical Conics.*

Correlatively, the system of concyclic conics of which $Y\Omega_1$, $Y\Omega_2$ are the cyclic arcs are transformed into concyclic conics of which $y\omega_1$, $y\omega_2$ are the cyclic arcs, the ellipses into ellipses, and the conics of the third species into conics of the third species; these denominations being again relative to the hemispheres which we are considering. (See M. Chasles "Sur les propriétés générales des coniques sphériques," art. 1—4). If D_1D_2, d_1d_2 are corresponding arcs, cutting the cyclic arcs in D_1, D_2, d_1, d_2, the spherical triangles D_1YD_2, d_1yd_2, in which $D_1Y = d_1y$, $D_2Y = d_2y$, supply the equations

$$\frac{\tan\frac{1}{2}(D_1+D_2)}{\tan\frac{1}{2}(d_1+d_2)} = \frac{\tan\frac{1}{2}(D_1-D_2)}{\tan\frac{1}{2}(d_1-d_2)} = \frac{\tan c}{\tan C}.$$

Let E, e be the areas of the spherical quadrilaterals $\Omega_1D_1D_2\Omega_2$, $\omega_1d_1d_2\omega_2$, we find $E = \pi - D_1 - D_2$, $e = \pi - d_1 - d_2$, whence

$$\frac{\tan\frac{1}{2}E}{\tan\frac{1}{2}e} = \frac{\tan C}{\tan c},$$

a formula which expresses a remarkable property of the cyclic arcs.

To chords of any conic (Φ) of the concyclic system of S, which cut off equal spherical areas from that conic, there will correspond chords cutting off from the corresponding conic (ϕ) areas equal to one another. To a spherical polygon of maximum area inscribed in any arc of (Φ), or to a polygon of minimum area circumscribing any arc of (Φ), there will correspond polygons possessing a similar maximum or minimum property with regard to the corresponding arc of (ϕ). These results follow from the known properties of concyclic spherical conics; or they may be deduced by reciprocation from the properties of the confocal conics of the two homographic systems.

(28.) *Arcs and Angles changed into equal Arcs and Angles.*

On any great circle A of S there are two points at right angles to one another, such that their corresponding points, on the corresponding great circle a, are also at right angles. These points are the external and internal points of bisection of the intercept made on the great circles by the cyclic arcs; they are also the points at which

the great circles are touched by conics of the concyclic systems. Let 2D, 2d be the intercepts; the homographic modulus of the two great circles (relative to the internal points of bisection) is $\frac{\tan \mathrm{D}}{\tan d}$; the arcs of the involution

$$\tan \mathrm{H}_1 \tan \mathrm{H}_2 = \frac{\tan \mathrm{D}}{\tan d}$$

are equal to the corresponding arcs of the involution

$$\tan h_1 \tan h_2 = \frac{\tan d}{\tan \mathrm{D}};$$

and the arcs of the involution

$$\tan \mathrm{H}_1 \tan \mathrm{H}_2 = -\frac{\tan \mathrm{D}}{\tan d}$$

are equal to the supplements of the corresponding arcs of the involution

$$\tan h_1 \tan h_2 = -\frac{\tan d}{\tan \mathrm{D}}.$$

The determination of the angles which at any point Δ are transformed into equal or supplementary angles at the point δ is correlative to the preceding. The external and internal bisectors of the angles between the radii vectores at Δ and δ are the right angles of the homographic pencils at A and a, and if $\mathrm{F}_1\Delta\mathrm{F}_2 = 2\Delta$, $f_1\delta f_2 = 2\delta$, the homographic modulus of the pencils, relative to the internal bisectors, is $\frac{\tan \Delta}{\tan \delta}$. The equiangular and supplementary involutions are respectively

$$\tan \mathrm{H}_1 \tan \mathrm{H}_2 = \frac{\tan \Delta}{\tan \delta}, \qquad \tan h_1 \tan h_2 = \frac{\tan \delta}{\tan \Delta},$$

and

$$\tan \mathrm{H}_1 \tan \mathrm{H}_2 = -\frac{\tan \Delta}{\tan \delta}, \qquad \tan h_1 \tan h_2 = -\frac{\tan \delta}{\tan \Delta}.$$

Combining the results relating to equal arcs and to equal angles, we see that, given any arc of a great circle in either figure, and a point upon it, there is always a spherical triangle having a vertex at the given point, and a second vertex upon the given arc, which is transformed into an equal and superposable spherical triangle.

The homographic modulus of the pencils at Δ and δ may be also expressed in terms of the radii vectores of the points Δ and δ, since from the triangles $\mathrm{F}_1\Delta\mathrm{F}_2$, $f_1\delta f_2$ we find

$$\frac{\tan \Delta}{\tan \delta} = \frac{\sin \frac{1}{2}(\mathrm{R}_1 - \mathrm{R}_2)}{\sin \frac{1}{2}(\mathrm{R}_1 + \mathrm{R}_2)} : \frac{\sin \frac{1}{2}(r_1 - r_2)}{\sin \frac{1}{2}(r_1 + r_2)}$$

$$= \frac{\cos \frac{1}{2}(\mathrm{R}_1 - \mathrm{R}_2)}{\cos \frac{1}{2}(\mathrm{R}_1 + \mathrm{R}_2)} : \frac{\cos \frac{1}{2}(r_1 - r_2)}{\cos \frac{1}{2}(r_1 + r_2)}.$$

(29.) *The Equations of the Homography in Spherical Coordinates.*

The equation of the cone Q, referred to its principal axes, is

$$\frac{\sin^2 C}{\sin^2 c} X^2 + Y^2 + \frac{\cos^2 C}{\cos^2 c} Z^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (Q);$$

the equations of its cyclic planes, and of its focal lines, are respectively

$$Y^2 + Z^2 \sec^2 c = 0,$$
$$Z^2 - X^2 \cot^2 c = 0,$$
$$X^2 + Y^2 \sin^2 c = 0;$$

and

$$Y^2 + Z^2 \cos^2 C = 0, \quad X = 0,$$
$$Z^2 - X^2 \tan^2 C = 0, \quad Y = 0,$$
$$X^2 + Y^2 \operatorname{cosec}^2 C = 0, \quad Z = 0.$$

The equations of the cone p, and of its cyclic planes and focal lines, are obtained by interchanging C and c.

Let Δ, δ be corresponding points on the two spheres, and let the arcs $X\Delta$, $Y\Delta$, $Z\Delta$, $x\delta$, $y\delta$, $z\delta$ meet the arcs YZ, ZX, XY, yz, zx, xy in the points A, B, C, a, b, c respectively. If we take the ratios of the cosines

$$X = \cos \Delta X, \quad Y = \cos \Delta Y, \quad Z = \cos \Delta Z,$$
$$x = \cos \delta x, \quad y = \cos \delta y, \quad z = \cos \delta z,$$

as the spherical coordinates of the points Ω and ω respectively, the homographic relation of the two figures is expressed by the equations

$$\frac{\sin C \cos X}{\sin c \cos x} = \frac{\cos Y}{\cos y} = \frac{\cos C \cos Z}{\cos c \cos z}.$$

Or again, if we take one of the following systems of tangents as the coordinates of the points Δ and δ,

(1) ..... $Y = \tan XB$, $Z = \tan XC$; $y = \tan xb$, $z = \tan xc$;
(2) ..... $Z = \tan YC$, $X = \tan YA$; $z = \tan yc$, $x = \tan ya$;
(3) ..... $X = \tan ZA$, $Y = \tan ZB$; $x = \tan za$, $y = \tan zb$;

the homographic relation is expressed by the equations

$$(1) \ldots\ldots Y = y \frac{\tan C}{\tan c}, \quad Z = z \frac{\sin C}{\sin c};$$

$$(2) \ldots\ldots Z = z \frac{\sin c}{\sin C}, \quad X = x \frac{\cos c}{\cos C};$$

$$(3) \ldots\ldots X = x \frac{\cos C}{\cos c}, \quad Y = y \frac{\tan c}{\tan C}.$$

(30.) *The Parameters of the Confocal and Concyclic Cones.*

Instead of the equation (Q), it will be convenient to employ the equation

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} + \frac{Z^2}{\Gamma^2} = 0$$

to represent the cone Q; so that

$$A : B : \Gamma :: \frac{\sin c}{\sin C} : 1 : \frac{\cos c}{\cos C},$$

$$\tan C = \sqrt{\left(\frac{B^2-\Gamma^2}{A^2-B^2}\right)}, \quad \tan c = \frac{A}{\Gamma}\tan C = \frac{A}{\Gamma}\sqrt{\left(\frac{B^2-\Gamma^2}{A^2-B^2}\right)}.$$

We suppose A, B, Γ all positive, and $A > B > \Gamma$, *i.e.*, $c > C$. The figure S is then transformed into s by the equations

$$(1) \quad \ldots\ldots\ldots\ldots X = Ax, \quad Y = By, \quad Z = \Gamma z,$$

or, writing $\alpha = \frac{1}{A}$, $\beta = \frac{1}{B}$, $\gamma = \frac{1}{\Gamma}$, by the equations

$$X = \frac{x}{\alpha}, \quad Y = \frac{y}{\beta}, \quad Z = \frac{z}{\gamma}.$$

We shall term the quantities Ψ and Φ the *parameters* of the confocal cone

$$(2) \quad \ldots\ldots\ldots\ldots \frac{X^2}{A^2-\Psi^2} + \frac{Y^2}{B^2-\Psi^2} + \frac{Z^2}{\Gamma^2-\Psi^2} = 0,$$

and of the concyclic cone

$$(3) \quad \ldots\ldots X^2\left(\frac{1}{A^2} - \frac{1}{\Phi^2}\right) + Y^2\left(\frac{1}{B^2} - \frac{1}{\Phi^2}\right) + Z^2\left(\frac{1}{\Gamma^2} - \frac{1}{\Phi^2}\right) = 0,$$

respectively. These quantities are of frequent use in the theory, as will appear from the following observations :—

(α.) If ψ and ϕ are the parameters of the cones corresponding to (Ψ) and Φ, we have $\psi = \frac{1}{\Psi}$, $\phi = \frac{1}{\Phi}$; for the cones (2) and (3) are transformed by the equations (1) into the cones

$$\frac{x^2}{\alpha^2-\psi^2} + \frac{y^2}{\beta^2-\psi^2} + \frac{z^2}{\gamma^2-\psi^2} = 0,$$

$$x^2\left(\frac{1}{\alpha^2} - \frac{1}{\phi^2}\right) + y^2\left(\frac{1}{\beta^2} - \frac{1}{\phi^2}\right) + z^2\left(\frac{1}{\gamma^2} - \frac{1}{\phi^2}\right) = 0,$$

of which the parameters are respectively ψ and ϕ.

(β.) If we imagine the principal axes of the two pencils coincident, the cone reciprocal to that confocal cone in S of which the parameter is Ψ is the concyclic cone in s of which the parameter is $\frac{1}{\Psi}$.

(γ.) If Ψ_1, Ψ_2 are the parameters of the two confocal spherical conics which pass through a given point, the parameter of the concyclic conic passing through that point is $\frac{AB\Gamma}{\Psi_1\Psi_2}$; and, reciprocally, if Φ_1, Φ_2 are the parameters of the concyclic conics touching a given arc, the parameter of the confocal conic touching that arc is $\frac{AB\Gamma}{\Phi_1\Phi_2}$.

(δ.) Let Λ, λ represent the focal semi-axes of the corresponding confocal conics (Ψ) and (ψ); we have

$$\sin^2\Lambda = \frac{\Psi^2-\Gamma^2}{A^2-\Gamma^2}, \quad \cos^2\Lambda = \frac{A^2-\Psi^2}{A^2-\Gamma^2}, \quad \tan^2\Lambda = \frac{\Psi^2-\Gamma^2}{A^2-\Psi^2},$$

with similar values for $\sin^2\lambda$, $\cos^2\lambda$, $\tan^2\lambda$; and hence

$$\sin\Lambda = \frac{\Psi}{A}\sin\lambda, \quad \cos\Lambda = \frac{\Psi}{\Gamma}\cos\lambda, \quad \tan\Lambda = \frac{\Gamma}{A}\tan\lambda.$$

(ϵ.) Thus, for the homographic modulus of the pencils at the corresponding points Δ, δ, we have the expression (see Art. 28)

$$\frac{\tan\Delta}{\tan\delta} = \frac{\sin\Lambda_2}{\sin\lambda_2} : \frac{\sin\Lambda_1}{\sin\lambda_1} = \frac{\Psi_2}{\Psi_1},$$

the angles being measured from the tangents to the confocals (Ψ_2), (ψ_2). And correlatively for the modulus of the homography on any corresponding arcs D, d, we have $\dfrac{\tan D}{\tan d} = \dfrac{\Phi_2}{\Phi_1}$,

the arcs D, d being measured from the points of contact of the concyclic conics (Φ_2) and (ϕ_2).

(ζ.) Lastly, if (Ψ_1), (Ψ_2) are the two confocals intersecting at Δ, (Φ) the concyclic conic passing through Δ, we have

$$\frac{\sin(\Lambda_1+\Lambda_2)}{\sin(\lambda_1+\lambda_2)} = \frac{\Psi_1\Psi_2}{A\Gamma}, \quad \text{or} \quad \frac{\sin R_1}{\sin r_1} = \frac{\sin R_2}{\sin r_2} = \frac{B}{\Phi},$$

an equation which corresponds to the equations (γ) of Art. 8.

(31.) *The Indicatrix on the Sphere.*

Let $d\Sigma_1$, $d\Sigma_2$, $d\sigma_1$, $d\sigma_2$ be corresponding elements of the spherical ellipses and spherical hyperbolas which pass through the corresponding points Δ and δ; let also the arcs Λ_1, Λ_2, λ_1, λ_2 be the focal semi-diameters of these conics; and let $2\Delta = F_1\Delta F_2$, $2\delta = f_1\delta f_2$. Considering two consecutive corresponding points on the two ellipses, and again on the two hyperbolas, we find

$$d\Sigma_1 = \frac{d\Lambda_2}{\sin\Delta}, \quad d\Sigma_2 = \frac{d\Lambda_1}{\cos\Delta},$$

$$d\sigma_1 = \frac{d\lambda_2}{\sin\delta}, \quad d\sigma_2 = \frac{d\lambda_1}{\cos\delta}.$$

But, differentiating the equations

$$\frac{\tan\Lambda_1}{\tan\lambda_1} = \frac{\tan\Lambda_2}{\tan\lambda_2} = \frac{\tan C}{\tan c},$$

we have $\dfrac{d\Lambda_1}{\sin 2\Lambda_1} = \dfrac{d\lambda_1}{\sin 2\lambda_1}$, $\dfrac{d\Lambda_2}{\sin 2\Lambda_2} = \dfrac{d\lambda_2}{\sin 2\lambda_2}$;

and from the triangles F_1PF_2, f_1pf_2,

$$\frac{\sin\Delta}{\sin\delta}=\frac{\cos C}{\cos\Lambda_1}:\frac{\cos c}{\cos\lambda_1}=\frac{\sin C}{\sin\Lambda_1}:\frac{\sin c}{\sin\lambda_1},$$

$$\frac{\cos\Delta}{\cos\delta}=\frac{\cos C}{\cos\Lambda_2}:\frac{\cos c}{\cos\lambda_2}=\frac{\sin C}{\sin\Lambda_2}:\frac{\sin c}{\sin\lambda_2};$$

whence

$$\frac{d\Sigma_1}{d\sigma_1}=\frac{\sin c}{\sin C}\times\frac{\sin\Lambda_1}{\sin\lambda_1}\times\frac{\sin 2\Lambda_2}{\sin 2\lambda_2},$$

$$\frac{d\Sigma_2}{d\sigma_2}=\frac{\sin c}{\sin C}\times\frac{\sin\Lambda_2}{\sin\lambda_2}\times\frac{\sin 2\Lambda_1}{\sin 2\lambda_1};$$

or, substituting from the equations (δ) and (γ), Art. 30,

$$\frac{d\Sigma_1}{d\sigma_1}=\frac{\Psi_1\Psi_2^2}{AB\Gamma}=\frac{\Psi_2}{\Phi},$$

$$\frac{d\Sigma_2}{d\sigma_2}=\frac{\Psi_2\Psi_1^2}{AB\Gamma}=\frac{\Psi_1}{\Phi}.$$

If in these formulæ we put $d\sigma_1=d\sigma_2=i$, the corresponding values of $d\Sigma_1$ and $d\Sigma_2$ are the principal semi-axes of the evanescent ellipse corresponding to the circle of which the centre is δ, and i the infinitesimal radius.

(32.) *Curves of Equal Tangential Deflexion and of Constant Elongation.*

Since $d\Sigma_1$ is the circular measure of the infinitesimal angle contained between the two lines in which (Ψ_1) is cut by (Ψ_2) and $(\Psi_2+d\Psi_2)$,

we have

$$d\Sigma_1^2=\frac{(\Psi_1^2-\Psi_2^2)\,\Psi_2^2\,d\Psi_2^2}{(A^2-\Psi_2^2)\,(B^2-\Psi_2^2)\,(\Psi_2^2-\Gamma^2)},$$

$$d\Sigma_2^2=\frac{(\Psi_1^2-\Psi_2^2)\,\Psi_1^2\,d\Psi_1^2}{(A^2-\Psi_1^2)\,(\Psi_1^2-B^2)\,(\Psi_1^2-\Gamma^2)},$$

which may also be deduced from the ordinary formulæ of elliptic coordinates in space. We may use these expressions to obtain the differential equations of certain loci analogous to those considered in Arts. 17, 18, and 19. Thus, observing that the homographic modulus of the pencil at Δ is $\frac{\Psi_2}{\Psi_1}$, we have for the curves of equal tangential deflexion the differential equation

$$(4)\ \ldots\ldots\ \frac{\sqrt{\Psi_2}\,.\,d\Psi_2}{\sqrt{[(A^2-\Psi_2^2)\,(B^2-\Psi_2^2)\,(\Psi_2^2-\Gamma^2)]}}=\frac{\sqrt{\Psi_1}\,.\,d\Psi_1}{\sqrt{[(A^2-\Psi_1^2)\,(\Psi_1^2-B^2)\,(\Psi_1^2-\Gamma^2)]}}.$$

The curves of "constant elongation" are defined by the equation

$$d\Sigma_1^2 + d\Sigma_2^2 = K^2 (d\sigma_1^2 + d\sigma_2^2),$$

or

$$\left(1 - \frac{K^2 \Psi_1^2 \Psi_2^4}{A^2 B^2 \Gamma^2}\right) d\Sigma_1^2 + \left(1 - \frac{K^2 \Psi_1^2 \Psi_2^4}{A^2 B^2 \Gamma^2}\right) d\Sigma_2^2 = 0,$$

in which the variables are not separated. If, however, we attend only to the curves of no elongation, and consider any tangent to one of them as determined by the parameters Φ_1 and Φ_2 of the two concyclic conics which it touches, its differential equation, in this system of tangential coordinates, is obtained by writing Φ_1 and Φ_2 for Ψ_1 and Ψ_2 in the equation (4). For, substituting $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, $\frac{1}{\psi_1}$, $\frac{1}{\psi_2}$ for A, B, Γ, Ψ_1, Ψ_2, in that equation, we have an equation between ψ_1 and ψ_2 of the same form as (4), which represents a curve of no elongation on the hemisphere s; ψ_1 and ψ_2 being the parameters of the concyclic conics which touch any tangent of that curve.

It is evident that an infinitesimal spherical area at any point of the concyclic conic (Φ) is altered in the ratio of $AB\Gamma : \Phi^3$; and, in particular, that this is the ratio of the area contained between (Φ) and ($\Phi + d\Phi$) on S, to the area contained between the corresponding curves (ϕ) and ($\phi + d\phi$) on s. The concyclic conics, as curves of constant alteration of adjacent infinitesimal areas, resemble the parallels to the vanishing line in the theory of two homographic plane figures.

(33.) *Circles of which a Focus is the Centre.*

Since, in general, spherical conics, of which F_1 or F_2 is a focus, are transformed into spherical conics of which f_1 or f_2 is a focus, and the director arcs of the corresponding curves are corresponding arcs, it follows that circles of which F_2 is the spherical centre, are transformed into conics of which f_2 is a focus, and of which the director arc is an arc $y\theta$ perpendicular to xz at a distance θ from x defined by the equation

$$\tan \theta = \frac{\tan c}{\tan^2 C}.$$

If R is the spherical radius of one of the given circles, and if r and δ denote the spherical distances from the focus and from the director arc of any point on the corresponding conic, we shall have the equation

$$\frac{\sin r}{\sin \delta} = \frac{\sin c}{\sin \theta} \times \frac{\tan R}{\tan C}.$$

Similarly, spherical conics, of which $Y\Omega_1$ or $Y\Omega_2$ is a cyclic arc, are transformed into spherical conics of which $y\omega_1$ or $y\omega_2$ is a cyclic arc; and the cyclic poles of corresponding curves are corresponding points. In particular, circles parallel to $Y\Omega_2$ are transformed into conics of which

$y\omega_2$ is a cyclic arc, and of which the cyclic pole is a point (ϕ) on xz at a distance φ from x defined by the equation

$$\tan\phi = \frac{\tan^2 c}{\tan \mathrm{C}}.$$

If R is the radius of a circle parallel to $\mathrm{Y}\Omega_2$, p the spherical perpendicular let fall from the cyclic pole on any tangent arc to the corresponding conic, ρ the angle contained between the tangent arc and the cyclic arc, we shall have the equation

$$\frac{\sin p}{\sin \rho} = \frac{\sin \phi}{\cos \mathrm{C}} \times \frac{\tan \mathrm{R}}{\tan c}.$$

The two arcs $y\theta$ may be termed the director arcs, and the two points (ϕ) the cyclic poles, of the figure s. It is evident that we shall have the relation $\tan\theta \tan\Phi = \tan\phi \tan\Theta = 1$, or $\Phi + \theta = \frac{1}{2}\pi = \phi + \Theta$.

(34.) *Circles changed into Circles.*

To determine the small circles of the sphere S which are transformed into small circles of the sphere s, we make use of the principle that a small circle of a sphere is a spherical conic having double contact with the imaginary asymptotic circle; the chord (or arc) of contact being the parallel great circle. Hence, the circles required are the spherical conics which have double contact with both P and Q. Of these circles there are three series corresponding to the three pairs of chords of intersection of P and Q. For the chords of contact of any one of the circles with P and Q are a pair of harmonic coujugates of one of the pairs of chords of intersection of P and Q; and, conversely, any such pair of harmonic conjugates may be taken for the chords of contact of a circle with P and Q, or again with Q and P. But the circles of only one of these series are real; their chords of contact being harmonic conjugates of the cyclic arcs, and their centres being on the great circle of the foci. Let R be the radius of one of these circles, Φ the distance of its centre (Φ) from X. The harmonic conjugate of the great circle, of which (Φ) is the spherical pole, with regard to the cyclic arcs, must have the same pole with regard to the imaginary conic Q and with regard to the circle. This condition supplies the equation

$$\tan^2 \mathrm{R} = -\frac{\sin(\Phi - \mathrm{C})\sin(\Phi + \mathrm{C})}{\sin(\Phi - c)\sin(\Phi + c)},$$

which determines the radius of the circle when the position of its centre is given; and shows that the circle is real only when Φ is intermediate between C and c.

If r be the radius of the corresponding circle, and ϕ' the distance of its centre from x, we shall have the equations

$$\frac{\tan R}{\tan r} = \frac{\sin 2C}{\sin 2c},$$

$$\tan \Phi \tan \phi' = \tan C \tan c.$$

The corresponding formulæ for the circles of the imaginary systems whose centres lie on XY, xy are

$$\tan^2 R = -\frac{\tan^2 C + \sec^2 C \tan^2 \Phi}{\tan^2 c + \sec^2 c \tan^2 \Phi},$$

$$\tan \Phi \tan \phi' = -\sin C \sin c,$$

$$\frac{\tan^2 R}{\tan^2 r} = \frac{\cos^2 c}{\cos^2 C} \cdot \frac{\tan^2 C}{\tan^2 c},$$

where Φ and ϕ' are the distances of the centres of the corresponding circles from X and x. Changing in these formulæ c and C into their complements, we have the formulæ for the corresponding imaginary circles of which the centres lie on ZY, zy.

(35.) *Theorems relating to Curvature.*

If two curves on either sphere touch one another at any point, the ratio of the tangents of their spherical radii of curvature remains unchanged in the transformation. This is evident from the corresponding theorem relating to plane homographical figures, because the two planes touching the spheres at two corresponding points are homographic. Thus the ellipses and hyperbolas on either sphere are lines of greatest or least alteration of curvature, as well as lines of greatest or least elongation or contraction. The circles which are transformed into circles are of course loci of points at which the tangent of the radius of curvature is altered in a constant ratio.

(36.) *Connexion with the Plane Theory.*

If we suppose the arcs C and c to become infinitely small, retaining a finite ratio to one another, the parts of the two spherical figures which lie infinitely near to X and x will ultimately become two plane similar figures. But we can also regard two dissimilar homographic plane figures as a limiting case of two homographic figures upon a sphere. The points of the two hemispheres, which we have hitherto considered, correspond to one another throughout the whole of each surface with similar directions of rotation. But if, in the hemisphere S, we substitute for the quadrant containing F_1 the opposite quadrant, so as to consider the hemisphere of which Z is the spheric centre, and the great circle XY the base, we shall obtain a figure of which one quadrant (F_2) answers with similar rotation to the corresponding quadrant (f_2), and the other quadrant (F_1) answers

with dissimilar rotation to the corresponding quadrant (f_1). If, for example, in the formulæ of Arts. 26—28, we change C, D, $\frac{1}{2}D_1$, or $\frac{1}{2}D_2$, H_1, H_2, Δ, $\frac{1}{2}R_1$, or $\frac{1}{2}R_2$, into their complements, we shall have the equations which express the metrical relations of the two figures, considered in this particular manner. In these new formulæ, $2c$ and $2C$ are the angles contained between the foci, and between the cyclic planes of s; or, again, they are the angles contained between the cyclic planes and the foci of S. The new arcs D, H_1, H_2, are not measured from the points corresponding to the original points of arcs d, h_1, h_2, but from points distant by a quadrant from the points corresponding to those original points; they are also measured backward — *i.e.*, in the direction opposite to that which corresponds to the direction in which the arcs d, h_1, h_2 are measured. And a correlative statement is true for the angles Δ, H_1, H_2. It will be observed that R_1 or R_2 is changed into its supplement according as the points considered lie in the regions of similar or dissimilar rotation. Again, it is immaterial whether we change D_2 or D_1 into its supplement; in the former case, we consider (in the figure S) the triangle D_1YD_2, in the latter the triangle $D_1\bar{Y}D_2$.

If we now suppose the arcs c and C to become evanescent, the parts of the two figures adjacent to x and Z respectively will become two dissimilar homographic plane figures, and we may pass from the spherical formulæ to the corresponding formulæ of the plane theory.

(37.) *Point-Figures in Perspective.*

When two homographic point-figures are in a perspective position, (*i.e.*, when the corresponding planes and lines of the two figures intersect upon the same plane,) one of the focal lines of each pencil is, evidently, the line joining the centres S and s of the two pencils. To find the other focal lines, let Ss meet the plane of intersection in O_1, let O be the point harmonically conjugate to O_1, with regard to Ss, and O_2 the orthogonal projection of O on the plane of intersection; SO_2, and sO_2, are the focal lines required.

To place two given homographic point-figures in a perspective position, we first of all place a pair of corresponding focal lines in the same straight line, the vertices of the two pencils not coinciding, but corresponding vectorial planes coinciding. Let O_2 be the point of intersection of the two remaining focal lines; let V_1 and V_2 be the planes which bisect the angle SO_2s externally and internally. According as the corresponding directions of rotation round SO_2 and so_2 are similar or dissimilar, V_1 or V_2 is the plane of intersection of the two homographic figures. It is evident that the two figures continue in perspective if their centres be moved nearer to or further from one

another in the coincident focal lines; or, again, if either of them be rotated through an angle of 180° round these coincident lines. Of the cyclic planes, one pair are parallel to the plane of intersection, the other pair intersect in that plane, and in the plane bisecting Ss at right-angles.

(38.) *Case when the Homography is Spheroidal.*

The theory of the particular case in which the transformation is *spheroidal—i.e.*, in which the imaginary cones Q and p, corresponding to the evanescent sphere-cones q and P, are cones of revolution—presents no difficulty whatever. If X and x are the centres of the imaginary small circles Q and p, the *azimuths* of any two corresponding points A and a are equal, and their *zenith-distances* are connected by the relation $\frac{\tan XA}{\tan xa} = \text{constant}$.

This constant ratio we may term the modulus of the transformation.

C.—Focal Properties of Two Homographic Spaces.

(39.) *The Imaginary Conics, and the Parameters.*

We proceed, in the last place, to consider two spaces S and s, homographically related to one another. Let Ω and σ be the imaginary circles at an infinite distance in which all spheres in the two spaces intersect one another; ω and Σ the imaginary conics corresponding to them. As we shall suppose that the planes at an infinite distance in the two spaces are not corresponding planes, the imaginary conics Ω and Σ, ω and σ, are certainly different. If either ω or Σ is an imaginary circle, the other is so too; for if Ω and Σ have a common chord, ω and σ must also have a common chord, and *vice versâ*. We shall, however, for the present, exclude this important particular case, and shall suppose that neither ω nor Σ is an imaginary circle. Let O, o' be the centres of Σ and ω respectively (these conics have no real tangents, and therefore are not parabolas); X, Y, x', y' the points at an infinite distance on their principal axes; Z, z' the points at an infinite distance on the normals to their planes; Σ_1, Σ_2, ω_1, ω_2 the asymptotic points of Σ, ω, lying on the lines XY, $x'y'$, which are the lines at an infinite distance in the planes of the two conics, and which we shall suppose to meet the imaginary circles Ω and σ in the points $\Omega_1\Omega_2$ and $\sigma_1\sigma_2$. The lines XY, $x'y'$ are evidently corresponding lines; and because the poles of XY, with regard to Ω and Σ, correspond to the poles of $x'y'$ with regard to ω and σ, the points Z, o' and the points O, z' are corresponding points. The anharmonic equation $[\Sigma_1, \Sigma_2, \Omega_1, \Omega_2] = [\sigma_1, \sigma_2, \omega_1, \omega_2]$, which is implied by the homographic relation of the figures, may also be written $[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2] = [\sigma_1, \sigma_2, \omega_1, \omega_2]$, and expresses, in this

form, that the imaginary angles $\Sigma_1 O \Sigma_2$, $\omega_1 \sigma' \omega_2$ are superposable; *i. e.*, that the imaginary conics Σ and ω are similar. Again, because X, Y are harmonic conjugates of $\Omega_1\Omega_2$ and $\Sigma_1\Sigma_2$, while x, y are harmonic conjugates of $\omega_1\omega_2$ and $\sigma_1\sigma_2$, x, y correspond to X, Y; and we may suppose the correspondence fixed by the equation

$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\omega_1, \omega_2, \sigma_1, \sigma_2, x, y].$$

This equation implies one or other of the equations

$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\sigma_1, \sigma_2, \omega_1, \omega_2, x, y],$$

or

$$[\Omega_1, \Omega_2, \Sigma_1, \Sigma_2, X, Y] = [\sigma_1, \sigma_2, \omega_1, \omega_2, y, x].$$

Of these, the former is inadmissible, as it would imply that $[\sigma_1, \omega_1, x, y] = [\omega_1, \sigma_1, x, y]$, which is impossible, since ω_2, and not σ_1, is the harmonic conjugate of ω_1 with regard to xy. We infer, therefore, that the point at infinity on the major axis of Σ corresponds to the point at infinity on the minor axis of ω, and *vice versâ*. Let $A\sqrt{(-1)}$, $B\sqrt{(-1)}$, $a\sqrt{(-1)}$, $b\sqrt{(-1)}$ be the principal semi-axes of Σ and ω; A, B are the parameters of S, and a, b of s; they are connected by the equation $Aa = Bb$, which results from the similarity of Σ and ω.

(40.) *The Correspondence of Directions—the Principal Axes.*

From the homographic relation of the two figures, it follows that to each direction on any straight line in either figure there corresponds a definite direction on the corresponding line. And again, to each direction of rotation round any line there corresponds a definite direction of rotation round the corresponding line. It is easily shown (by considering in each figure two infinitesimally near positions of a straight line in relation to a line at a finite distance) that the two figures are either similar in respect of all rotations, or dissimilar in respect of all rotations; *i.e.*, that corresponding rotations round corresponding directions are either always similar, or else always dissimilar. For clearness we may suppose that corresponding rotations in the two figures are similar. We shall call the lines OX, OY, OZ, $o'x$, $o'y$, $o'z$ the principal axes of the two figures; and the planes OYZ, OXZ, OXY, $o'yz$, $o'xz$, $o'xy$ the principal planes; the axes OZ, $o'z$, which alone are corresponding lines, we shall call the focal axes; and the planes of XY, xy the vanishing planes.

Each space is divided by its three principal planes into eight octants, corresponding respectively to the eight octants of the other space. Considering these octants as tetrahedra, of which the plane at an infinite distance is one boundary, and observing that in either space the plane at an infinite distance corresponds to the vanishing plane of the other space, we find that to adjacent octants on the same side of the vanishing plane in either space there correspond in the other space adjacent octants on the same side of the vanishing plane, but that adja-

cent octants on opposite sides of the vanishing plane in either space correspond to octants diametrically opposite in the other space; so that if the correspondence of two octants is given, that of the remaining octants is immediately ascertained. Again, if P, Q are any two points on the same side of the vanishing plane of S, and if p, q are the points corresponding to P, Q, the directions PQ, pq are corresponding directions in the two spaces; and similarly the corresponding directions of rotation round any two corresponding lines may be ascertained. We may add that if V is any closed figure in S, which lies wholly on one side of the vanishing plane, points in the interior of V will correspond to points in the interior of a closed figure v corresponding in the space s to V.

(41.) *Determination of the Principal Axes and Parameters.*

The geometrical construction for the determination of the principal axes in each figure, and of the parameters A, B, a, b, is as follows. We first obtain the vanishing plane of each figure; *i. e.*, we determine in each figure three points corresponding to three points at an infinite distance in the other figure; the points at an infinite distance in the directions normal to the vanishing planes are the points Z and z', and the points corresponding to these are the centres o' and O of the imaginary conics ω and Σ; thus the focal axes OZ, and $o'z'$ are known. At the point O, in the vanishing plane of S, take two pairs of lines corresponding to two pairs of rectangular lines intersecting at z' in the plane at an infinite distance in s. The axes OX and OY are the pair of lines at right angles to one another in the involution determined by the two pairs so constructed; x and y, which determine $o'x$ and $o'y$, are the points corresponding to X and Y. Lastly, to find the parameters, we observe that if in any two corresponding planes the chords intercepted by Σ and ω are $2D\surd(-1)$, and $2d\surd(-1)$ respectively, the parameters of the two homographic plane figures are D and d; their homographic centres are the points of bisection of the chords, and their focal axes are the perpendiculars to the chords at their points of bisection. Hence we obtain the four parameters A, B, a, b by constructing the homographic foci of the principal planes XZ, YZ, xz, yz.

(42.) *The Confocal Quadrics.*

The imaginary conic Σ, in which we may suppose $A > B$, determines a system of confocal quadrics, of which it is the imaginary focal conic. The two real focal conics are an ellipse in the plane of YZ, of which the foci (in the axis of Z) are the homographic foci of the plane YZ, and of which the vertices, in the same axis, are the homographic foci of the plane XZ. The focal hyperbola lies in the plane of XZ, and has of course the vertices of the ellipse for foci, and its foci for vertices. The system

of confocal quadrics of which ω is the imaginary focal conic, correspond homographically to the confocal quadrics of the system S. For since the conics σ and ω correspond to the conics Σ and Ω, the imaginary developable circumscribing the two former conics corresponds to the imaginary developable circumscribing the two latter conics, and therefore the quadrics inscribed in these corresponding developables are themselves corresponding surfaces. In particular, to the focal ellipse of S there corresponds the focal hyperbola of s, and *vice versâ*; the extremities of the focal axes of the ellipses being transformed into the extremities of the focal axes of the hyperbolas, and the extremities of the minor axes of the ellipses into the asymptotic points of the hyperbolas. Again, the ellipsoids of either confocal system are changed into the hyperboloids of two sheets of the other system; and the hyperboloids of one sheet into the hyperboloids of one sheet. And by considering the two pairs of homographic planes XZ, xz, YZ, yz, we see immediately that the eccentricities of the sections of corresponding confocals made by corresponding principal planes are reciprocal, and that the rectangle of their major semi-axes is equal to the rectangle of the parameters $A \times a$ or $B \times b$. Again, to the normals of any confocal there correspond the normals of the corresponding confocal; the lines of curvature of the two surfaces, their umbilics, the two systems of orthogonal developables formed by the normals of each of them, their centres of curvature, and the surfaces which are the loci of those centres, all correspond homographically; the cuspidal lines of the normal developables are corresponding geodesic lines upon the surfaces of centres, and the lines of contact of two corresponding developables with those sheets of the surfaces of centres upon which their cuspidal lines do not lie, are in like manner corresponding lines. Further, since the normals of corresponding confocals are corresponding lines, the geodesics of either surface correspond to the geodesics of the other; and the confocals enveloped by the developables of two corresponding geodesics are corresponding confocals, and the lines of contact are corresponding lines. To the various modes of description of the lines of curvature of either system of confocals by means of a thread stretched upon surfaces of the system, there will correspond similar modes of description of the lines of curvature of the other system of confocals. For an example, we may take the general theorem of M. Chasles,

"If an inextensible thread, of which the extremities are fastened to two fixed points upon one of two confocal surfaces of different kinds, is strained by the point of a pencil which moves upon the second surface, so that the thread consists (in general) of six portions, two of which are geodesics of the first surface, two are geodesics of the second surface, while the other two are the portions of common tangents to the two surfaces included between the points of contact, the

point of the pencil will describe a line of curvature of the second surface."

While an inextensible thread moves in either space in the manner described in this enunciation, an inextensible thread will move in the same manner in the other space; and the six portions of the first thread will correspond homographically to the six portions of the second. But it is to be observed that the constant lengths of the two threads will be related transcendentally to one another; as also will the lengths of the corresponding curvilinear portions of the two threads.

We may add that to two geodesic arcs of which the difference is rectifiable, there will correspond two geodesic arcs of which the difference is rectifiable. And when the difference of two arcs of a line of curvature can be expressed by geodesic lines in either figure, the corresponding difference can be similarly expressed in the other figure.

(43.) *The Point-Figures at Corresponding Points—their Focal Lines.*

We shall next consider any two corresponding points P and p in the two spaces. At these two points we have two homographic point-figures, of which the relations to one another are readily ascertained. To the cones which from P envelope the conics Ω and Σ, there will correspond the cones which from p envelope ω and σ. Thus the principal axes of the point-figures at P and p are the normals to the surfaces of the confocal system which pass through P and p; and the focal lines of the figures are the generators of the hyperboloids of a single sheet which pass through P and p. We thus have the theorem:

"Any two corresponding generators of two hyperboloids of the two confocal systems are the axes of pencils of planes of which the correspondence is equiangular."

If the points P and p be taken on corresponding focal conics, the two generators coincide. Thus, "the focal conics are the loci of points at which the correspondence of the homographic point-figures is spheroidal."

It is evident that, given in one of the two spaces a point and three generators (of the same or different hyperboloids), and the corresponding things in the other space, we can immediately, by means of the equiangular pencils of planes, determine the point p in either space which corresponds to a given point P in the other. We might take for the three generators in each space any three tangents to a focal conic; the simplest construction being perhaps that in which the tangents at the vertices of the focal conics are employed as the axes of equiangular pencils.

We thus obtain the following rule, which is well adapted to the methods of descriptive geometry: — "Project the given point P

orthogonally on the planes of XZ, YZ, and using the focal radii vectores of the projections, as in Art. 5, determine the points corresponding to them in the planes of xz and yz: these points are the orthogonal projections of the point p."

(44.) *The Strain Ellipsoid—its Cyclic Planes and Focal Asymptotes.*

The position of the cyclic planes of the homographic figure at P may be ascertained by means of the focal lines of the figure at p. But these cyclic planes are also the cyclic planes of the "strain ellipsoid" at P; *i.e.*, of the evanescent ellipsoid which has its centre at P, and corresponds to an evanescent sphere having its centre at p. For this evanescent ellipsoid has for its asymptotic cone the imaginary cone which from P envelopes Σ, and is thus concentric, similar, and similarly situated with the auxiliary ellipsoid of M. Chasles, *i.e.*, with the ellipsoid of which the principal axes are equal to the axes major of the three confocal surfaces passing through P, and are normal to those three surfaces respectively (Aperçu historique des Méthodes en Géométrie, Note 25). It appears at the same time that the asymptotes of the focal conic of the auxiliary ellipsoid, or of the strain ellipsoid, coincide with the focal lines of the point P.

The cyclic planes at P and p are the "planes of no distortion" at those corresponding points; *i.e.* (1) evanescent lines passing through P and lying in either cyclic plane, are altered in a constant ratio; (2) angles in a cyclic plane at P are transformed into equal angles in the corresponding cyclic plane; so that P, p are homographic foci of either pair of cyclic planes. The second property is analogous to the property that the focal lines are the axes of equal homographic pencils of planes. If we observe that the focal asymptotes of a quadric are the axes of its circumscribing right cylinders, we may enunciate a property of the focal lines analogous to the first property of the cyclic planes:—

"Planes parallel to either focal line, and infinitely near to P, are transformed into planes, which may ultimately be regarded as parallel to the corresponding focal line, and of which the distances from p are in a constant ratio to the distances of the first planes from P."

We may express this by saying that a generating line of a confocal hyperboloid is, at any point of it, a line of equal transverse elongation. And since the right cylinder, of which the focal line at P is the axis, and which circumscribes the strain ellipsoid at P, is transformed into a right cone of which the vertex lies on the vanishing plane of s, we see that if the point P vary its position on a given hyperbolic generator, the ratio of transverse elongation varies inversely as the distance of p from the vanishing plane of s, or directly as the distance of P from the vanishing plane of S.

(45.) *The Canonical and Elliptic Equations.*

If we represent by X, Y, Z, x, y, z, the coordinates of corresponding points in the two spaces referred to their principal axes, the canonical equations of the homography will be

$$Zz = Aa = Bb$$

$$\left.\begin{aligned} Xz &= Ax \\ Yz &= By \end{aligned}\right] \text{ or } \left.\begin{aligned} xZ &= aX \\ yZ &= bY \end{aligned}\right] \ldots\ldots\ldots\ldots\ldots\ldots\text{(A).}$$

If, again, we denote the elliptic coordinates of corresponding points in either space (referred to the corresponding confocal systems) by $\Lambda_1, \Lambda_2, \Lambda_3, \lambda_1, \lambda_2, \lambda_3$, the homographic equations are

$$\Lambda_1\lambda_1 = \Lambda_2\lambda_2 = \Lambda_3\lambda_3 = Aa = Bb \ldots\ldots\ldots\ldots\ldots \text{(B)};$$

so that every general homographic transformation may be represented as a transformation of the elliptic coordinates of a point into their reciprocals.

(46.) *Determination of the Strain Ellipsoid.*

Either of these sets of formulæ will serve to determine the ratios of the axes of the strain ellipsoid at P to the radius of the evanescent sphere at p. The rectangular formulæ show that the ratio of an evanescent volume at P to the corresponding volume at p is that of Z^4 to $Aa \times ab$; whence, if $\theta\Lambda_1$, $\theta\Lambda_2$, $\theta\Lambda_3$ are the semi-axes of the strain ellipsoid at P, and i the radius of the evanescent sphere at p,

$$\theta^3 \frac{\Lambda_1\Lambda_2\Lambda_3}{i^3} = \frac{Z^4}{Aa \times ab},$$

or, since $\Lambda_1\Lambda_2\Lambda_3 = ABZ$, and $Zz = Aa$, $\theta = \dfrac{i}{z}$.

Or again, transforming by the equations (B) the elliptic formula

$$d\Sigma_1 = \sqrt{\left[\frac{(\Lambda_1^2 - \Lambda_2^2)\,(\Lambda_1^2 - \Lambda_3^2)}{(\Lambda_1^2 - A^2)\,(\Lambda_1^2 - B^2)}\right]}\, d\Lambda_1,$$

we find

$$d\sigma_1 = \frac{Aa \times AB}{\Lambda_1\Lambda_2\Lambda_3}\,\frac{d\Sigma_1}{\Lambda_1},$$

or

$$\frac{d\Sigma_1}{\Lambda_1} = \frac{d\sigma_1}{z},$$

which agrees with the preceding determination of θ, the symbols $d\Sigma_1$ and $d\sigma_1$ representing corresponding elementary arcs, normal to (Λ_1) and (λ_1).

Our limits prevent us from applying these formulæ to the determination of the loci corresponding to those considered in Arts. 17—19. For the same reason, we omit the elementary theorems relating to the curvature and torsion of curve lines, and the curvature of curve surfaces.

(47.) *The Parameters of the Confocal and Concyclic Cones, at any Point.*

The equation of the imaginary cone which from the point P envelopes Σ is

$$\frac{X^2}{\Lambda_1^2}+\frac{Y^2}{\Lambda_2^2}+\frac{Z^2}{\Lambda_3^2}=0,$$

and the cone which from the same point envelopes the confocal surface, of which Ψ is the semi-axis major, is

$$\frac{X^2}{\Lambda_1^2-\Psi^2}+\frac{Y^2}{\Lambda_2^2-\Psi^2}+\frac{Z^2}{\Lambda_3^2-\Psi^2}=0;$$

so that the coefficients (designated by A, B, Γ, α, β, γ in Art. 30), which determine the homography of the point-figures at P and p, are, in fact, the elliptic coordinates of those points; and the parameters of the confocal cones of the point-figures are the same as the parameters of the confocal quadrics which they envelope. Thus, the formulæ of Arts. 30, 31 are immediately applicable to the figures at P and p. And, if Φ is the parameter of a concyclic cone at P, so that $\Phi=\frac{\Lambda_1\Lambda_2\Lambda_3}{\Psi_1\Psi_2}$, where Ψ_1,Ψ_2 are the parameters of two confocal quadrics touching any line of (Φ), the elongation at P in the direction of any line of (Φ) is given by any one of the formulæ

$$\frac{T}{\tau}=\frac{\Lambda_1\Lambda_2\Lambda_3}{Aa\times AB}\Phi=\frac{\Lambda_1^2\Lambda_2^2\Lambda_3^2}{Aa\times AB}\times\frac{1}{\Psi_1\Psi_2}=\frac{Z\Phi}{Aa}=\frac{B}{a}\times\frac{Z^2}{\Psi_1\Psi_2}=\frac{\Phi}{z},$$

T and τ representing corresponding elements at P and p. It will be observed that, at equal distances from the vanishing plane, the elongation is the same on all lines touching the same two confocal quadrics.

(48.) *Lines Tangent to two Confocal Quadrics.*

Let L_1 and L_2 be two straight lines in the space S, each of which touches the two confocals (Ψ_1) and (Ψ_2); let also l_1, l_2, (ψ_1), (ψ_2) be the corresponding lines and confocal quadrics in the space s. The tangent planes $L_1\Psi_1$, $L_1\Psi_2$ (*i. e.*, the tangent planes to (Ψ_1), (Ψ_2), at their points of contact with L_1) are at right-angles to one another, and are transformed into two planes which are at right-angles to one another. Again, the pair of planes, tangent to any third confocal surface (Ψ_3), which intersect in L_1, make the same angles with the bisecting planes $L_1\Psi_1$, $L_1\Psi_2$ that the pair of planes, tangent to the same confocal surface (Ψ_3), and intersecting in L_2, make with the bisecting planes $L_2\Psi_1$, $L_2\Psi_2$. For the involutions of pairs of planes determined by the confocal system at the lines L_1 and L_2 are necessarily equiangular in respect of

all their corresponding pairs, because they are equiangular in respect of the coincident pairs of planes determined by (Ψ_1) and (Ψ_2), and of the imaginary pair of cyclic planes determined by the imaginary circle at an infinite distance. From this theorem (of which M. Chasles has given a different demonstration; see Liouville, Vol. XI., First Series, p. 109) we infer that equal dihedral angles, similarly placed in the pencils of planes at L_1 and L_2, are transformed into dihedral angles equal to one another, and placed similarly to one another, in the pencils of planes at l_1 and l_2. Or again, if I and i are the angles made with $L_1\Psi_1$ and $l_1\psi_1$ by corresponding planes passing through L_1 and l_1, and if we denote the major semi-axes of the surfaces (Ψ_1), (Ψ_2), (ψ_1), (ψ_2) by Ψ_1, Ψ_2, ψ_1, ψ_2, we shall have the equation

$$\frac{\tan I}{\tan i} = \frac{\Psi_1}{\Psi_2} = \frac{\psi_2}{\psi_1},$$

which results immediately from a formula given by M. Chasles (*loc. cit.* p. 106), combined with the equations of transformation (B); and which shows, in conformity with our theorem, that the ratio of tan I to tan i is the same, whatever common tangent of (Ψ_1) and (Ψ_2) we consider. We have, in fact, the still more general theorem:

"All pencils of planes, of which the axes are touched by two confocals having their major semi-axes in a given ratio, have that ratio for their modulus of transformation; and in all such pencils, the involutions which are transformed into equiangular involutions, are equiangular with one another," which is an immediate consequence from Art. 47, and 30, ϵ.

Since a generating line of a confocal hyperboloid may be regarded as a line of which the two tangent confocals coincide, this enunciation includes, as a particular case, the equiangular property of the generating lines.

We have seen that the focal conics are the loci of points at which the transformation is spheroidal. We may now add, that at any one of these points the modulus of transformation (Art. 38) is $\frac{Q}{P}$, if Q is the semi-axis major (A or B) of the focal conic on which the point is taken, and P is the semi-axis major of the confocal quadric which passes through the point.

(49.) *Ivory's Theorem.*

If on two confocal surfaces of the same kind in the space S we consider two points which *correspond* to one another in the sense in which that term is employed in Ivory's theorem, these two points will be transformed into two others in the space s, which will also *correspond* to one another in the same sense. This principle, which is immediately verified by means of the equations (A), may serve to transform some

geometrical, and even physical, propositions. For example, we see that to every focal generation of a quadric according to Jacobi's method, there corresponds homographically a similar focal generation of another quadric.

(50.) *Equi-Segmental Axes and Planes.*

The equi-segmental axes of all planes in the space S, which cut the vanishing plane in straight lines parallel to a given straight line, (or, which is the same thing, of all planes which pass through a given point at an infinite distance on the vanishing plane,) lie on two planes at equal distances from the vanishing plane and parallel to it.

For, in the first place, parallel planes in the space S have their equi-segmental axes at one and the same distance from the vanishing plane, since to parallel planes in the space S there correspond planes in the space s, which intersect the vanishing plane of that space in the same straight line, and of which the foci are consequently at a constant distance from one another; this constant distance being equal to the distance of the equi-segmental axes of the planes in the space S. Again, planes in the space S, which intersect the vanishing plane in the same straight line, have their equal axes situated at equal distances from the central plane. For to these planes correspond parallel planes in the space s; and, by what has just been proved, the equi-segmental axes of these planes lie in two planes parallel to the vanishing plane; therefore the equi-segmental axes of the planes in S lie in two corresponding planes, *i. e.* in two planes parallel to the vanishing plane. The theorem itself results from the combination of these two particular cases of it.

It may be worth while to verify the theorem analytically. If

$$\frac{X}{p} + \frac{Y}{q} + \frac{Z}{r} = 1$$

is the equation of any plane of S, the equation of the corresponding plane of s is

$$\frac{Ax}{pz} + \frac{By}{qz} + \frac{Aa}{rz} = 1,$$

which meets the vanishing plane of s in the line

$$z = 0, \quad \frac{Ax}{p} + \frac{By}{q} = -\frac{Aa}{r},$$

or

$$z = 0, \quad \frac{x}{a\frac{p}{r}} + \frac{y}{b\frac{q}{r}} = -1.$$

The square of the semichord determined on this line by the imaginary conic ω, or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0,$$

is $$-\frac{(a^2p^2+b^2q^2)\,(q^2r^2+p^2r^2+p^2q^2)}{r^2\,(p^2+q^2)^2};$$

and this square, multiplied by the square of the sine of the angle between the given plane and the vanishing plane of S, becomes

$$-\frac{a^2p^2+b^2q^2}{p^2+q^2},$$

an expression of which the value depends only on the ratio of p to q.

It follows from this theorem, that to ascertain the position in the space S of the equi-segmental axes of any plane whatever, it will suffice to attend to the principal equi-segmental axes, *i.e.* to the equi-segmental axes of planes which pass through the focal axis. Let P be such a plane, and let $D\sqrt{(-1)}$ be the semi-diameter of the imaginary focal conic lying in that plane, $d\sqrt{(-1)}$ the semi-diameter of the imaginary focal of s, determined by the corresponding plane. It will be found that $Dd=Aa$; so that we have for the semi-distance d of the equi-segmental axes of the plane P, the expression $d=\frac{Aa}{D}$. Thus all the planes, loci of real equi-segmental lines, or, as we shall term them, all the equi-segmental planes of S, are comprised between two planes, at distances a and b on the positive side of the vanishing plane, and between two planes symmetrically situated on the negative side of the same plane. Again, since the semi-diameters of S, which are equally inclined to its principal axes, are equal to one another, each equi-segmental plane contains two distinct series of equi-segmental parallels, the two series being equally inclined to the plane of ZX, or ZY; in the two extreme pairs of equi-segmental planes these two series coincide with one another, and their common direction is that of one of the principal axes OX or OY.

If we consider two planes intersecting in the axis of Z, and inclined at an angle I to the plane of ZX, we have for the square of the semi-distance of their equi-segmental lines the expression,

$$d^2=a^2\cos^2 I+b^2\sin^2 I.$$

The corresponding inclination i is given by the equation

$$\tan I=\frac{B}{A}\tan i=\frac{a}{b}\tan i,$$

and the corresponding value of D^2 is

$$D^2=A^2\cos^2 i+B^2\sin^2 i.$$

These equations show that if we imagine the spaces S and s so placed that their focal axes coincide, while the axes of X and Y lie in the axes of y and x respectively, the principal equi-segmental axes of either space will be those generating lines of hyperboloids of the other space, which lie in planes parallel to the central plane.

The homographic relation of any two corresponding equi-segmental planes is very simple. If we conceive of the points of each plane as referred to its principal equi-segmental axes, the corresponding co-ordinates of corresponding points will be equal, and only the angle between the axes will be different in each plane. These angles are never equal to one another (except in the excluded case $A=B$, $a=b$); they are, however, supplementary to one another in the *principal* equi-segmental planes, *i.e.* in the planes defined by the equations $Z = \pm\sqrt{(ab)}$, $z = \pm\sqrt{(AB)}$, since in these planes we have

$$\tan^2 I = \frac{B}{A}, \quad \tan^2 i = \frac{b}{a}, \quad \tan I \tan i = 1.$$

To obtain the straight line in s, which corresponds to any given straight line in S, we may either determine its projections on the two focal planes, by means of the equi-segmental axes in those planes; or we may, instead, consider the intersections of the given straight line with any pair of equi-segmental planes of S, and obtain the corresponding points in the corresponding equi-segmental planes of s. We have, however, in every case to measure the equal corresponding segments in corresponding directions; and these can always be ascertained by inspection, if we have first fixed the correspondence of the eight octants of each space to the eight octants of the other.

(51.) *Properties of the Hyperbolic Generators.*

The generating lines of the confocal hyperboloids possess a metrical property with regard to the equi-segmental planes, which may be very variously expressed, according to the equi-segmental planes considered. Thus:

"The intercept made in the space S upon any generator of a confocal hyperboloid by the tangent planes to that hyperboloid, which are parallel to the vanishing plane, is to the corresponding intercept in the space s in the constant ratio of $\sqrt{(AB)}$ to $\sqrt{(ab)}$."

Or, again:

"The intercept made on any hyperbolic generator of S by the two equi-segmental planes $Z = \pm\sqrt{(ab)}$, is to the corresponding intercept in the space s in the inverse ratio of the major-axes of the hyperboloids to which the two generators belong."

In connexion with this property we may mention the following; which, however, does not depend on the general homographic transformation we are considering:—

"If one of two confocal hyperboloids be transformed into the other by the transformation of Ivory, segments on any generator of the one are transformed into equal segments on the generator of the other."

(52.) *Homographic Spaces placed Symmetrically.*

It is in general impossible to place two homographic spaces S and s in the same space, so that any given point of that space shall have the same corresponding point, to whichever of the two spaces it is considered to belong. The conditions that this reciprocal relation of the two spaces should be possible are, that corresponding rotations in the two figures should be similar, and that

$$(C) \ldots\ldots A = b, \quad B = a,$$

either of these equations, of course, implying the other. For, if these equations be satisfied, and if corresponding rotations be similar, we may place the axes of OZ, OX, OY upon the axes of oz, $\pm oy$, $\pm ox$, inasmuch as the positive directions of OZ and oz are not corresponding directions. Writing, as we may then do, $\pm$ X for Y and $\pm$ Y for X in the equations (A), we find

$$zZ = ab, \quad zX = \pm by, \quad zY = \pm ax,$$

and these equations are not altered by interchanging simultaneously X, x; Y, y; Z, z. The points which coincide with their conjugates are the points of the lines

$$z = +\sqrt{(ab)}, \quad y = \pm\sqrt{\left(\frac{a}{b}\right)}\, x,$$

$$z = -\sqrt{(ab)}, \quad y = \mp\sqrt{\left(\frac{a}{b}\right)}\, x;$$

i.e., the principal equi-segmental planes of S coincide with their corresponding planes, and in each of these planes the points of one of the principal equi-segmental axes coincide with their corresponding points. Every plane which passes through either of these lines, corresponds to itself, and so does every line which meets both of them. Again, we may also place the axes of OZ, OX, OY upon the axes of $-oz$, $\pm oy$, $\mp ox$; in this case, the equations (A) become

$$zZ = -ab, \quad zX = \mp by, \quad zY = \pm ax,$$

which are still symmetrical, but which give imaginary loci of coincident points. Either the upper signs, or else the lower signs, may be taken in each case; so that the two spaces admit of four different symmetrical positions.

We may arrive at the preceding results without using the equations (A); for it is readily seen that the necessary and sufficient conditions for the reciprocity of the two homographic systems are that the imaginary conics Σ and ω should coincide, and that those points on the two conics should be coincident, which correspond to the same points of the imaginary circle at an infinite distance. The equations (C) are the conditions that the two conics should be equal in all respects; if these equations are satisfied, the two conics can be brought into coincidence in four different ways, and in each of these four ways the points which ought to coincide will coincide, if corresponding rotations in the two spaces are similar.

(53.) *Case of a Spheroidal Homography.*

It is hardly necessary to do more than mention the case of a spheroidal homography, in which $A=B$, $a=b$. All meridian planes of the space S have the same foci at a distance $\pm A$ from the equatorial (or vanishing plane), and their equi-segmental axes lie in the same two parallel planes at a distance $\pm a$ from the equatorial plane.

The angle contained by any two meridian planes is unchanged in the transformation; and the homographic relation is the same for all pairs of corresponding meridian planes. Thus, all angles between planes and lines intersecting at either focus remain unchanged in the transformation, and the pencils in space at corresponding foci are superposable. Similarly, each equi-segmental plane is superposable upon the plane corresponding to it. The two spaces may, in fact, be conceived as generated by the equiangular rotation of two homographic planes round their focal axes. The condition that they should be capable of occupying a reciprocal position, is that the distances between the foci in each space should be equal.

(54.) *Historical Note.*

The existence of two pairs of parallel equi-segmental axes in any two homographic plane figures was established by M. Moebius in 1827. ("Barycentrische Calcul," p. 320, sect. 230.) M. Moebius also showed that, if the corresponding points of two corresponding equi-segmental axes coincide in the line of intersection of two homographic planes, the two planes are in perspective. Magnus ("Sammlung von Aufgaben und Lehrsätzen aus der Analytischen Geometrie," Berlin, 1833, p. 41, sect. 12) proved that in two homographic plane figures there exists a pair of corresponding points at which the corresponding pencils are equiangular; and that, if the figures be placed in the same plane with these "centres of collineation" coincident, and either of them rotate in its own plane round the centre of collineation, it will become homological with the other in two diametrically opposite positions, in one of which positions one pair of equi-segmental axes will coincide, while the other pair will coincide in the other position. Magnus expressly says that "of two collinearly-related systems" [*i.e.*, two homographic plane figures in which the straight lines at an infinite distance are not corresponding lines] "each has, in general, only one centre of collineation." As Magnus tacitly supposes that the figures are not in any position whatever with regard to one another, but are already placed in the same plane, this statement is not untrue; but it is only part of the truth, and the analysis by which Magnus obtains one centre of collineation in each figure, will also supply a second pair, if we change the sign of the constant p in the equations (1) of p. 42

loc. cit. It is of course quite true that if the two figures are once placed in the same plane, there is only one point in each which can be regarded as a centre of collineation; and this, which Magnus has proved analytically, Dr. Salmon has also shown geometrically ("Higher Plane Curves," Art. 230, p. 246). But it is to be remembered that two planes can be made to coincide in two different ways according as they are placed face to face, or both facing the same way, and, in one of these positions of coincidence, one of the pairs of foci are the centres of collineation, and the other pair in the other position. It is worth while to add that though, as Dr. Salmon has observed, the position of the imaginary circular points at an infinite distance is unaffected by any motion of translation or rotation of a plane figure in its own plane, those two imaginary points are interchanged with one another if the figure be rotated, through an angle of 180°, round any axis in its own plane. And the change of the centre of collineation, which takes place when one of two homographic figures, of which the planes are coincident, is thus rotated, is a necessary consequence of the interchange of the imaginary cyclic points in the rotated figure.

In the "Traité de Géométrie Supérieure," only one pair of equi-segmental axes and one pair of foci are expressly mentioned. But the omission is only accidental, as the methods by which one pair of foci and one pair of equi-segmental axes are obtained would equally supply the other pair. The theorem, that "if two planes are in perspective, the foci are the points in which they are intersected by the perpendiculars let fall from the centre of perspective on the planes bisecting the angles contained by the two planes," is an immediate inference from a principle, first given by M. Chasles ("Aperçu de l'Histoire des Méthodes en Géométrie," note iv.), and subsequently employed by Mr. Mulcahy ("Principles of Modern Geometry," cap. VIII., art. 115).

Subsequently to the communication of this memoir to the London Mathematical Society, but (it is unnecessary to say) quite independently of it, three papers have appeared, relating in part to the same subject. (1) In the May number of the "Nouvelles Annales de Mathématiques," M. Abel Transon obtains the theorem of the two pair of foci by the application of a very general analytical method; he accurately describes the similarity and dissimilarity of the foci, and speaks of the theorem itself as "une propriété de l'homographie qui n'avait peut-être pas encore été remarquée." (2) M. Richelot, of Königsberg, in a paper dated Oct. 29, 1868, and published in the second part of the 70th volume of Crelle's Journal, has considered the analytical theory of homographic figures in space, and has been led to the consideration of their focal properties. It would seem, however, that M. Richelot supposes the tangents of the focal conics to be the only axes of equiangular pencils of planes; whereas, as we have shown, this property is possessed by every generating line of any confocal hyperboloid.

The cause of the oversight (if it is one) appears in the words : "Es muss, in der That, eine Axe im obigen Sinne [i.e., if we understand M. Richelot correctly, a line which is the axis of a pencil of planes equiangular with its corresponding pencil] die Eigenschaft besitzen, dass unter den unendlich vielen auf ihr senkrechten Ebenen eine existirt, deren entsprechende Ebene auf der der Axe entsprechenden Gerade senkrecht steht" (p. 141). This property, however, is not possessed by every axis of a pencil of planes equiangular with its corresponding pencil, but only by those which lie in one of the principal planes. M. Richelot speaks of a forthcoming work of a pupil of his own, M. Maegis, as intended to contain a complete analytical theory of homography in space. (3) In the November number of the Nouvelles Annales de Mathématiques, M. Housel enuntiates the theorem : "En deplaçant sans déformation deux figures homographiques dans l'espace, on peut les rendre homologiques." This theorem is not in accordance with Art. 50 of the present Paper, because in that article we have in effect shown that corresponding equi-segmental planes are never superposable except in the case of a spheroidal homography. But the analysis of M. Housel seems insufficient to establish his conclusion, since it is not shown that the values ultimately obtained of the ten unknown quantities of Art. XIII. of M. Housel's Memoir actually satisfy the twelve equations of that article. [The values of the unknown quantities are not obtained in an explicit form, and there are only ten of them, and not eleven, because p depends on X, Y, Z.] And, considered in itself, the conclusion is inadmissible ; for any homological transformation of space must change the imaginary circle, in which all spheres intersect, into a circle, whereas in general that circle is changed into an imaginary ellipse by a homographic transformation. Again, the homographic relation depends on fifteen constants, the homological relation on seven, and the six constants of displacement can only reduce the fifteen constants to nine. Thus it would seem *à priori* that two conditions must be satisfied in order that two homographic spaces should be capable of a homological position. And the equation $A=B$ (or $a=b$) of Art. 50 is equivalent to two independent relations connecting the fifteen constants of the homography, since that equation is equivalent to the two conditions that a certain conic should be a circle.

Note.—The focal properties of homographic point-figures might be obtained by simple considerations of perspective (see Arts. 21 and 37). We have, however, preferred to deduce these properties from their genuine source—the properties of the imaginary circle in which all spheres intersect one another at an infinite distance. In the case of homographic plane figures, we have ventured to employ both methods successively (Arts. 1—4, and Art. 5). This has been done at some risk of repetition; but it seemed desirable to exhibit this part of the theory in its most elementary and practical, as well as in its most abstract form, in the hope that some of the simpler results may be found of use in the actual practice of perspective.

APPENDIX.

In accordance with the wishes of the Council, the second volume of the Society's Proceedings closes with the present Number. An attempt has been made to recover the shorter, and in many cases very interesting, communications which have from the outset been given in the course of discussions on the main papers of the Evening Meetings. A few of these are appended below, and references are, in some cases, given to the works in which others subsequently appeared.

Taking these communications in the order in which they were made, we have "The regular Hypocycloidal Tricusp," by M. Jenkins, B.A., (read June 19th, 1865); a portion of this paper is given in the "Educational Times" for September, 1865.* "A proof of Euclid i. 47 not involving the definition of a parallelogram," communicated by A. De Morgan, F.R.A.S., (Nov. 20th, 1865). This proof, which is given in many German editions of Euclid, is now also to be met with in Cassell's Elements. "On Motion in a Circle, and its relation to Planetary Motion," communicated by Prof. Sylvester, (Dec. 18th, 1865,) has since been given in "Nugæ Mathematicæ," extracted from the "Philosophical Magazine for 1866."

At the same meeting, Prof. Cayley gave the following simple method for finding the volume of a Tetrahedron. "If a, b be the lengths of two opposite sides, λ their inclination, h the length of their shortest distance; the section by a plane perpendicular to the line h, at a distance x from the side b, is a parallelogram, angle λ and sides $\frac{x}{h}a$, $\frac{h-x}{h}b$, whence element of volume is

$$\frac{x}{h}a \cdot \frac{h-x}{h}b \sin\lambda\, dx = \frac{ab\sin\lambda}{h^2}x(h-x)\,dx;$$

whence integrating from $x = 0$, to $x = h$, the whole volume is

$$\frac{ab\sin\lambda}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \tfrac{1}{6}abh\sin\lambda."$$

"The Centres of Algebraical Curves and Surfaces," by S. Roberts, M.A., (read March 19th, 1866,) was subsequently printed in the "Quarterly Journal of Mathematics," Vol. IX., No. 33, p. 25.

A fuller account of Prof. Sylvester's paper, read at the same meeting, of which an abstract is given in No. VI., appears in the "Proceedings

* Mathematical Reprint, Vol. IV., p. 58.

of the Royal Society," No. 84, 1866, where it is entitled, "On the Motion of a Rigid Body moving freely about a fixed point."

At the meeting held April 16th, 1866, Prof. Cayley called attention to the theorem, that the difference between two consecutive prime numbers may exceed any given number N−1 whatever. For if $a, b, c \ldots k$ are the prime numbers not greater than N, then $abc \ldots k+1$, and $abc \ldots k+1+N$ may be one or both of them prime, but all the intermediate numbers are composite; that is, the difference of the two successive primes is = N at least. Mr. A. J. Ellis, at the same meeting, communicated the following constructions:—"From a pair of conjugate diameters in an ellipse, to find the foci and the axes." Let CP, CD be the conjugate semi-diameters. Through P draw MPN, bisected in P, twice the length of CD, and perpendicular to it. Draw CS bisecting the angle MCN, and a mean proportional between the lengths of CM, CN. S is one focus, whence H the other, and directions of the axes are given. Draw PN, DR perpendicular to CS, then CA, CB being the semi-axes, their lengths are given by the equations

$$CA^2 = CN^2 + CR^2,\ CB^2 = PN^2 + DR^2.$$

If, by the conjugate diameter CD in an hyperbola, we mean the diameter of the conjugate hyperbola which is parallel to the tangent at the extremity P of the first diameter, and preserve the same letters, the construction is the same, with these differences; MPN must be parallel to CD (instead of being perpendicular to it), and $CA^2 = CN^2 \sim CR^2$, $CB^2 = PN^2 \sim DR^2$. This construction is easier and more complete than that given for the ellipse only, in Chasles' "Sections Coniques, art. 205." The demonstration depends upon Proceedings of the Royal Society, 14th June, 1866, vol. 15, p. 200, equations d, e. (where in equation d, read $o'e$ for $o'e'$,) first given by Mr. Ellis, and having a much wider application. A solution of the problem, "Given a pair of conjugate diameters of an ellipse, to find any number of points on the curve," by R. Tucker, M.A., (communicated Nov. 22nd, 1866,) has since appeared in the "Mathematical Reprint," Vol. VII., p. 28. "Proof of the Rectangle of Forces," by J. J. Walker, M.A., (communicated March 28th, 1867,) was given subsequently in the "Quarterly Journal of Mathematics," Vol. IX., No. 34, p. 173.

It remains only to add a few words of explanation about the two Indices. They are intended to be mutually exclusive. The first gives the names of all the authors of papers, and nearly all their communications; the second is concerned with points which turn up in the communications themselves.

A short list of the Errata which have been detected follows; doubtless others exist. The Secretaries will be glad to receive a list of any which have escaped their notice, for insertion in a future Volume.

R. T.

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† This paper, extracted from the Admiralty Manual for the Deviations of the Compass, Third Edition, was presented by the author to those gentlemen who were members of the Society at the date of its being read (Dec. 10th, 1868).

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ERRATA.

xvii. p. 137, last line of *text*, *for* "when the radius of the circle is unity," *read* "when the radius of the generating circle is unity divided by the factorial of n."

xviii. p. 175, lines 4 and 7 from bottom, *for* "ALKANTRI" *read* "ALKAUHI."

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